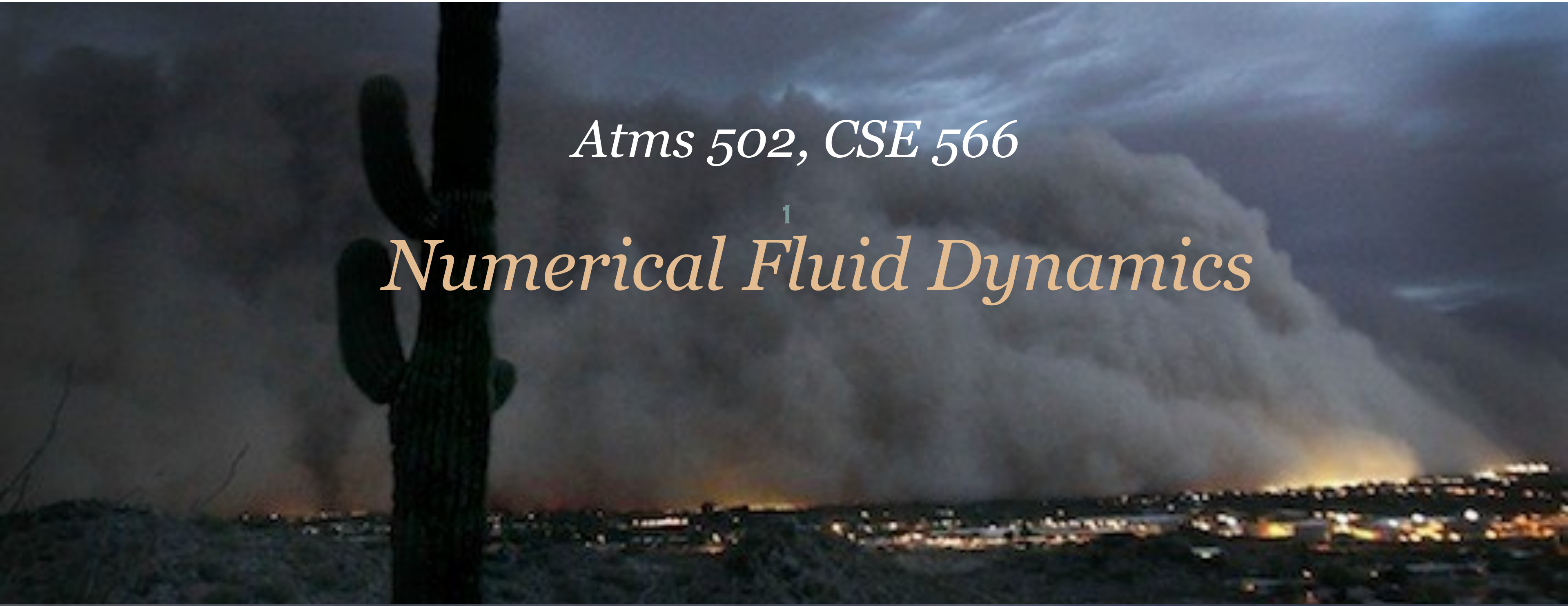


Atms 502, CSE 566

Numerical Fluid Dynamics

Thu., Apr. 4, 2019

ATMS 502
CSE 566

Thursday,
4 April 2019

Class #22

Program #5 is due Tuesday, April 16

Plan for Today

- 1) Review *Semi-Lagrangian methods*
- 2) Staggered grids
  - Why we bother
- 3) Diffusion; Advection+diffusion
  - Filtering vs. integrating; stability
- 4) Program 5
  - continued
- 5) Lee & Wilhelmson
  - Intro: Density currents, 3D model

Staggered grids

WHY WE BOTHER.

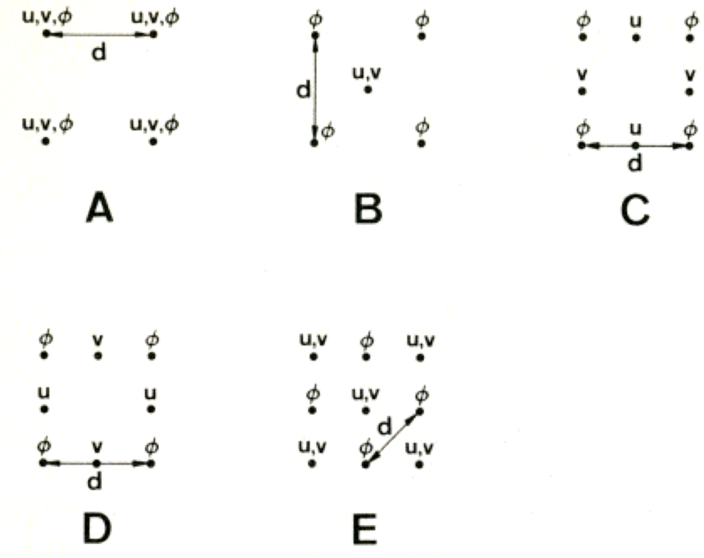


Figure 7.4 The five grids that were analyzed by Winninghoff (1968) and Arakawa and Lamb (1977).

Staggered grids require a smaller time step

Staggered grid stability

Shallow water:

- Unstaggered:

$$\delta_{2t}u + U\delta_{2x}u + g\delta_{2x}h = 0$$
$$\delta_{2t}h + U\delta_{2x}h + H\delta_{2x}u = 0$$

✓ Result: $$\left(|U| + c\right)\frac{\Delta t}{\Delta x} < 1$$

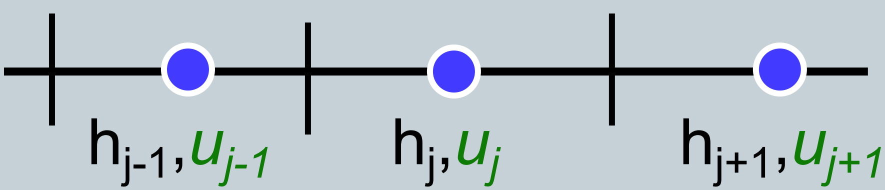
Shallow water:

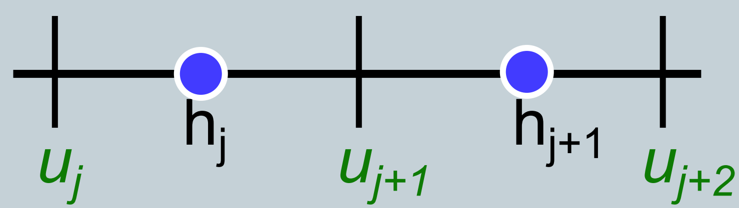
- Staggered:

$$\delta_{2t}u + g\delta_{x}h = 0$$
$$\delta_{2t}h + H\delta_{x}u = 0$$

✓ Result: $$\left|\frac{c\Delta t}{\Delta x}\right| < \frac{1}{2}$$

Accuracy: both order $(\Delta x)^2$; staggered case error less





Staggered grid phase

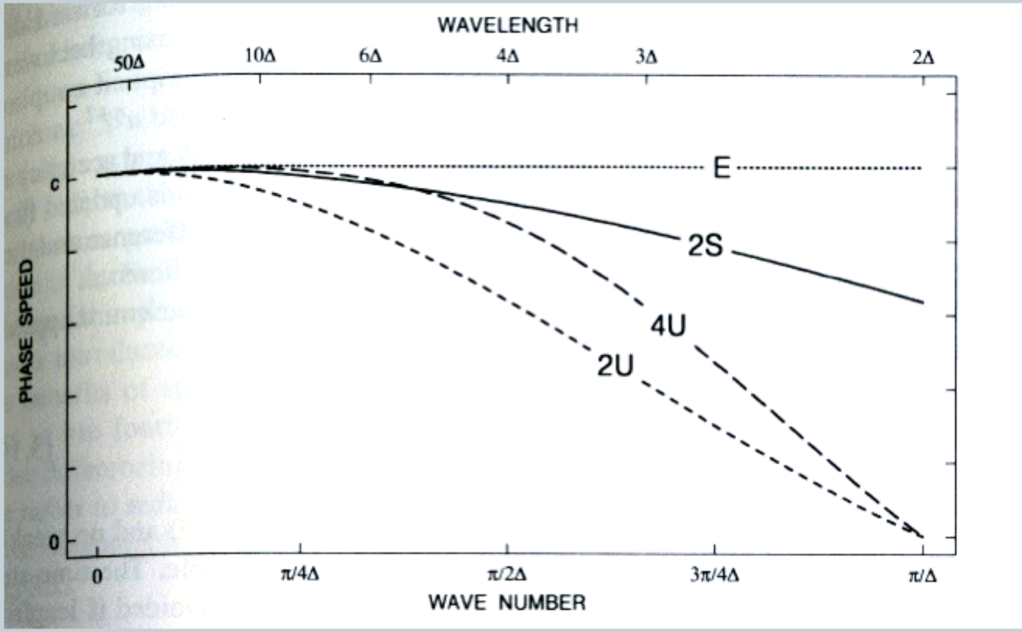
Phase speed analysis

$$c_{unstag} = \frac{\omega}{k} = \frac{c}{k\Delta x} \sin k\Delta x$$

unstaggered

$$c_{stag} = \frac{\omega}{k} = \frac{2c}{k\Delta x} \sin \frac{k\Delta x}{2}$$

staggered



- Staggered 2nd order better than 2nd *or 4th* order unstaggered.
- Group velocity: unstaggered 2x worse

Diffusion - and the advection-diffusion problem

Diffusion: two 3-point-filter approaches

- Using a 1-step filter
  - A simple 1D, 1-pass, 3-point filter: $\bar{\phi} = (1-S)\phi_j + \frac{S}{2}\left(\phi_{j+1} + \phi_{j-1}\right)$
  - The *response function*:
    - S=1/2: $2\Delta x$ eliminated
    - S<0: *desmoothing*

Let $\phi(x) = Ae^{ikj\Delta x}$ Then, $\bar{\phi} = R\phi$,

with R = response function $= 1 - 2S\sin^2\frac{k\Delta x}{2}$

- Using diffusion in a PDE

$$\frac{d\phi}{dt} = \gamma\left(\phi_{j+1} - 2\phi_j + \phi_{j-1)}\right) = \gamma(Ae^{ikj\Delta x})\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right) =$$

$$= -2\gamma Ae^{ikj\Delta x}(1-\cos k\Delta x) = \frac{dA}{dt}e^{ik(j\Delta x)}$$

$$\frac{dA}{dt} = -2\gamma(1-\cos k\Delta x)A \Rightarrow R \equiv \frac{1}{A}\frac{dA}{dt} = -2\gamma(1-\cos k\Delta x)$$

γ = damping coefficient
A = amplitude
R = response value;
$2\Delta x$ is most damped.

Diffusion: exact solution, amplification factor

- Exact solution
  - Starting with: $\phi_t = M\phi_{xx}$
  - An exact solution: $\phi(x,t) = \phi_0 e^{ikx} e^{-Mk^2 t}$
  - The exact amplification error:

$$A_e = \frac{\phi(t+\Delta t)}{\phi(t)} = e^{-Mk^2\Delta t} = e^{-r\Delta x^2 k^2} = e^{-r\beta^2};\ r = \frac{M\Delta t}{\Delta x^2}$$

    - smallest waves are damped the most!

Diffusion: exact vs. simple FTCS approach

- Forward time, centered space: $$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t} = M\left(\frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2}\right)$$
- Behavior: $$\lambda = 1 - 4\nu \sin^2\left(\frac{\beta}{2}\right); \quad \nu = M\Delta t / \Delta x^2$$
  - Damps 2Δx most
  - stable for $0 < \nu \leq ½$
    - But $\nu > ¼$ has $\lambda < 0$ (sign flips) for 2Δx
    - To avoid 'over damping' use $0 < \nu \leq ¼$

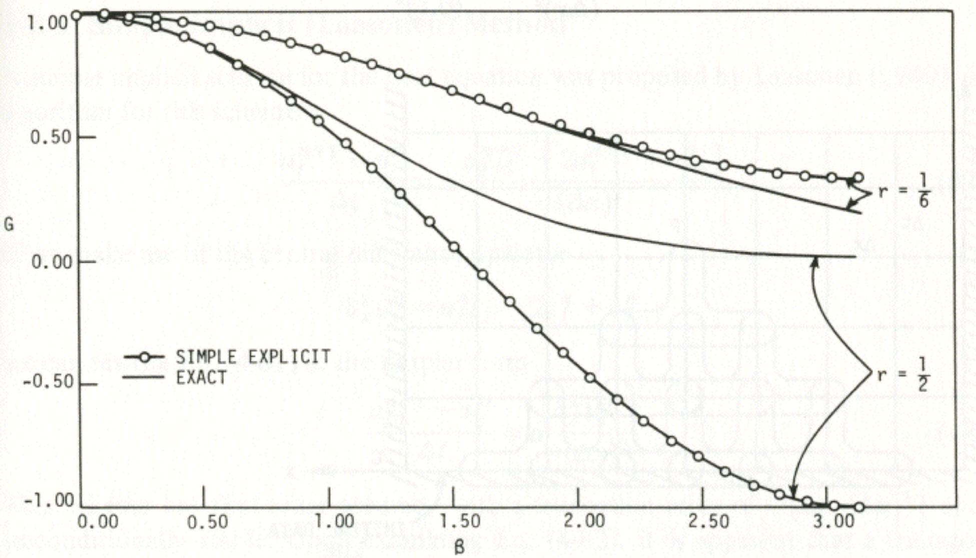


Figure 4-12 Amplification factor for simple explicit method.

ANDERSON ET AL. 1984

General diffusion, and some conclusions

- General form: $$\frac{\phi_j^{n+1}-\phi_j^n}{\Delta t} = M\frac{\theta\delta_{xx}\phi_j^{n+1}+(1-\theta)\delta_{xx}\phi_j^n}{\Delta x^2} \text{ with } 0 \le \theta \le 1$$
  - $\theta=0$: simple explicit; $\theta=1$: simple implicit; $\theta=½$: Crank-Nicolson
  - Accuracy: $\theta=0,1$: $O(\Delta t,\Delta x^2)$; $\theta=½$: $O(\Delta t^2,\Delta x^2)$
- Stability and accuracy: advection vs. diffusion
  - Advection problem:
    - Small wavelengths unstable first; may propagate away and/or amplify, impacting overall solution
  - Diffusion problem:
    - Smallest wavelengths damped the most
    - Durran: accuracy of treatment of smallest waves is “irrelevant” if they are decaying anyway..

Advection + diffusion problem

WE ARE *TRYING* TO SOLVE:

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = M\frac{\partial^2 \phi}{\partial x^2}$$

- *Diffusion must dominate!*
  - Upstream advection, centered diffusion
    - Modified equation:

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = M\left(1 + \frac{P_e}{2}\right)\frac{\partial^2 \phi}{\partial x^2} \quad where \quad P_e = \frac{c\Delta x}{M}$$

  - Centered advection, centered diffusion
    - Modified equation:

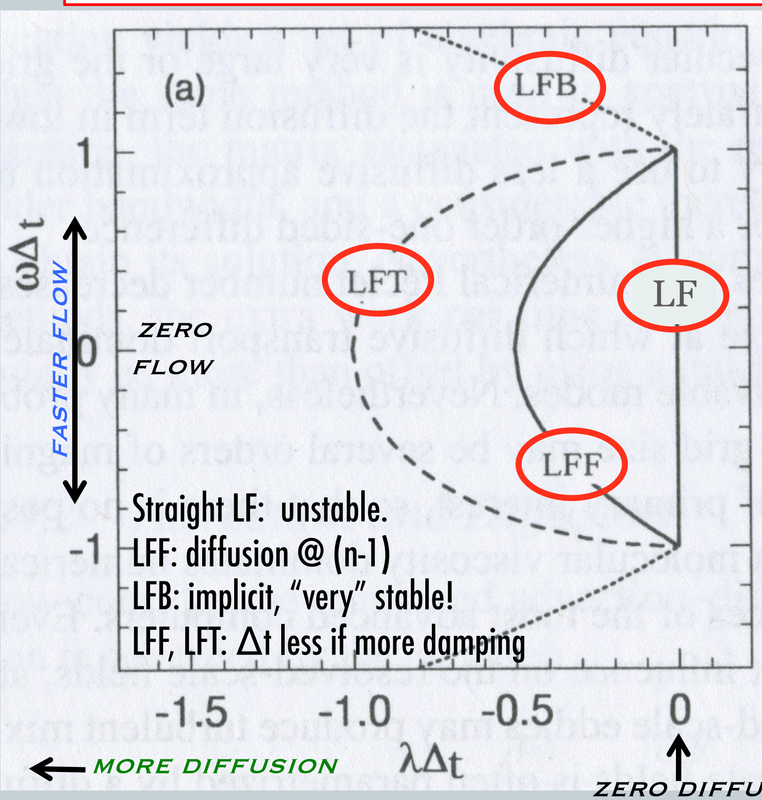
$$\phi_t + c\phi_x = \left(M - \frac{c^2 \Delta t}{2}\right)\phi_{xx} + ()\phi_{xxx} + ()\phi_{xxxx} + \ldots$$

- Summary:
  - The *Péclet number* must remain small, or the solution is no longer consistent. Small P_e requires
    - small flow speed *c*
    - small grid spacing Δx, or
    - large damping coefficient *M*
      - M may be prognostic, and so vary throughout the flow.

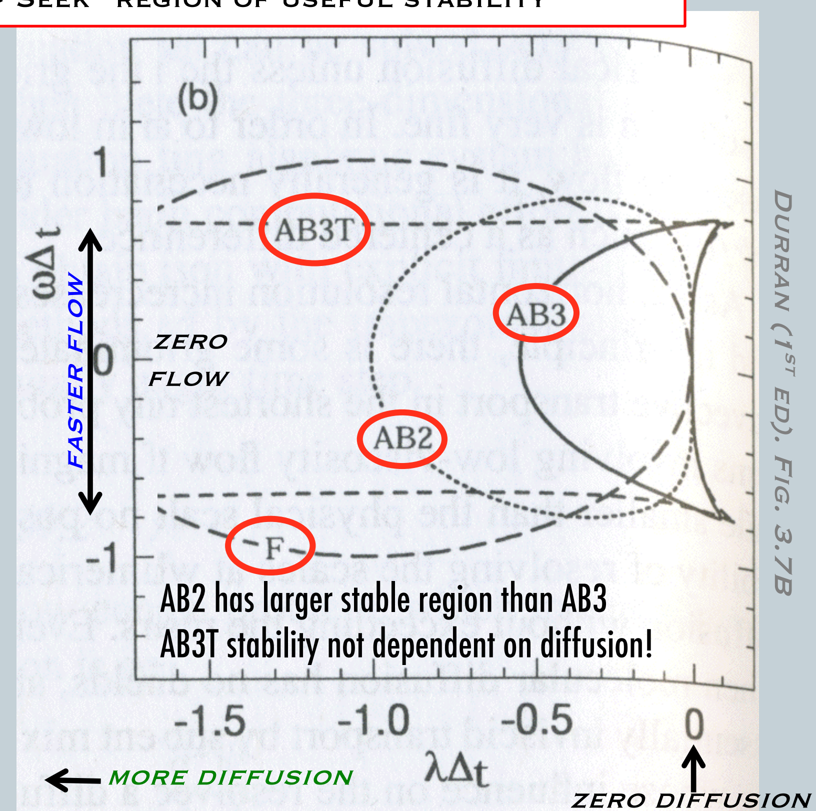
Advection+diffusion: Time differencing

- Leapfrog time differencing
  - Straight LF, LFF (forward) *as in P6*
- Adams-Bashforth stability
  - 2^{nd}, 3^{rd}-order, trapezoidal

AXES ARE SCALED ADVECTION/DIFFUSION COEFFICIENTS. >> SEEK "REGION OF USEFUL STABILITY"



DURRAN (1^{ST} ED). FIG. 3.7A



DURRAN (1^{ST} ED). FIG. 3.7B

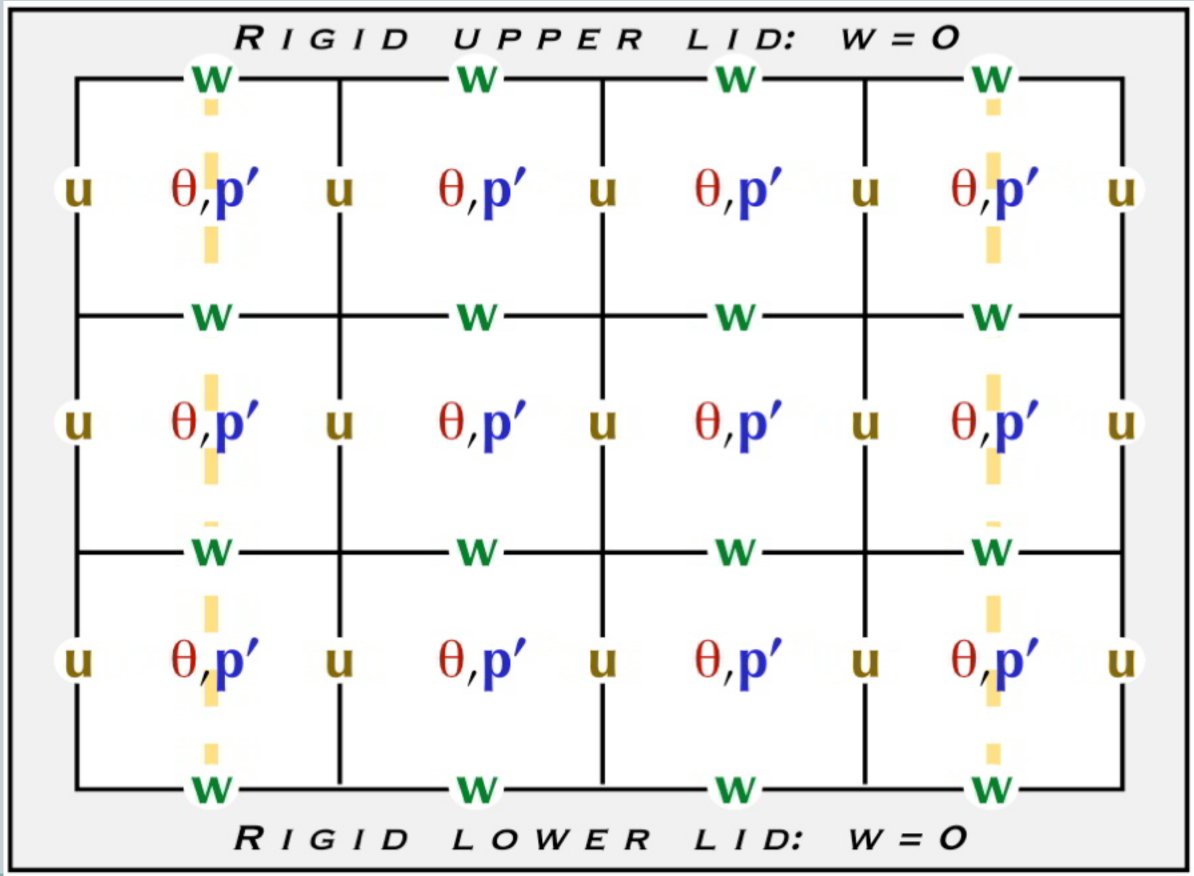
$$\tilde{\omega} = \omega\Delta t = -kv\Delta x = -\beta v; \quad \tilde{\lambda} = \lambda\Delta t = \left(-Mk^2\right)\Delta t$$

Program #5

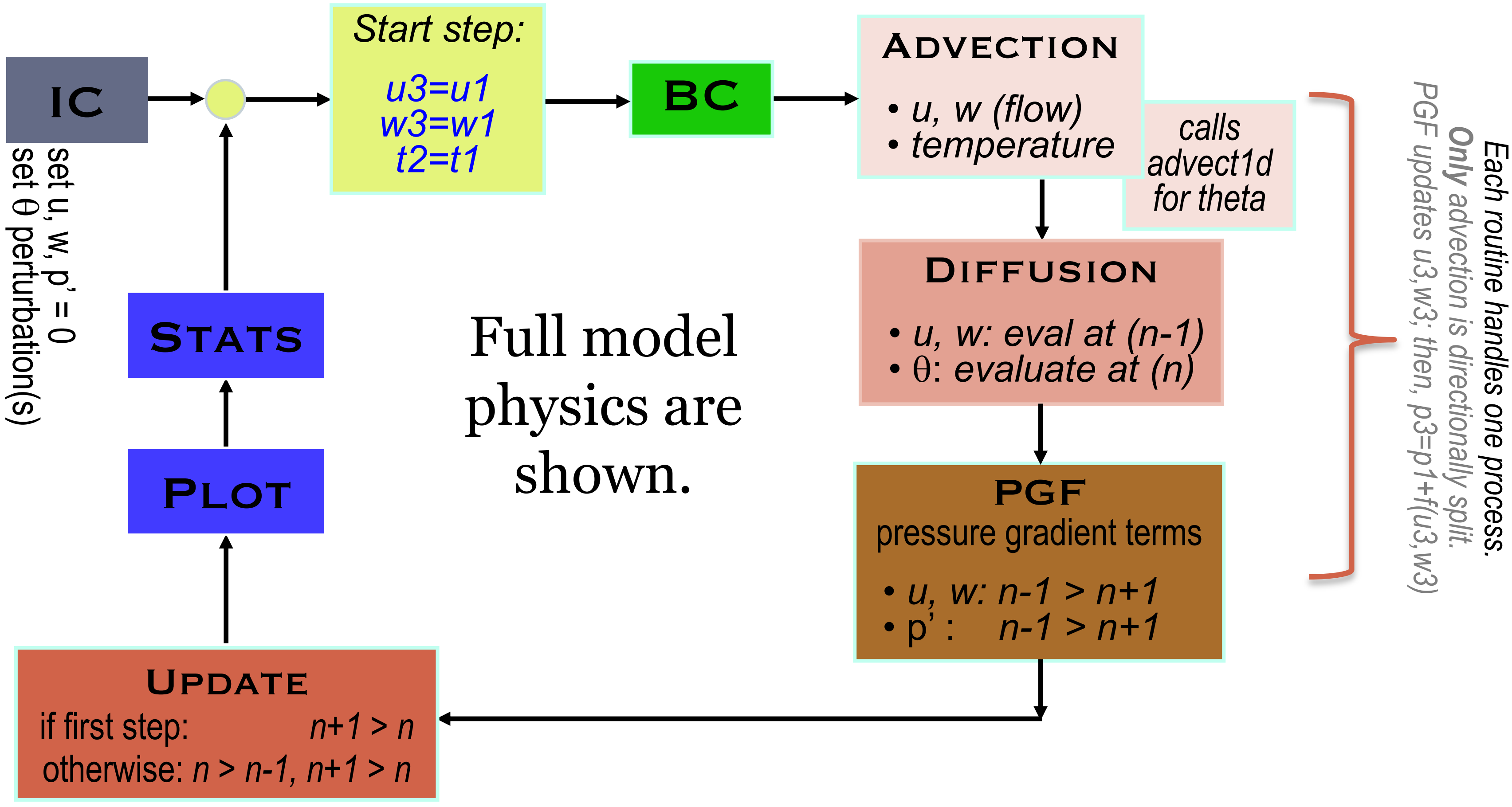
2D NONLINEAR, COMPRESSIBLE FLOW

Program 5: BCs

- Boundary conditions (B.C.'s)
  - Z: B.C.'s as before
    - *zero-gradient*
    - *(w: zero @ top, bottom)*
  - X: BCs
    - **symmetric** B.C.'s
      - for W, P, θ
    - **asymmetric** in X
      - *only* for U



Program 5: Structure



Program 5: Coding practice

- Starting a time step
  - Before doing anything else:
    - t2 = t1
    - u3 = u1
    - w3 = w1
    - All later routines *add to* these "n+1" arrays.

This also lets us turn processes on or off – we have taken the 'first part' of each time step – before starting.

  - So in *advection, diffusion, PGF, you will code …*
    - $t2 = t2 + \ldots \Delta t \bullet [\text{ forcing terms }]$
    - $u3 = u3 + \ldots 2\Delta t \bullet [\text{ forcing terms }]$ *(same for W)*
  - Exception: pressure
    - Only one step to pressure: $p3 = p1 + 2\Delta t \bullet [\text{ forcing terms }]$

Program 5: First time step

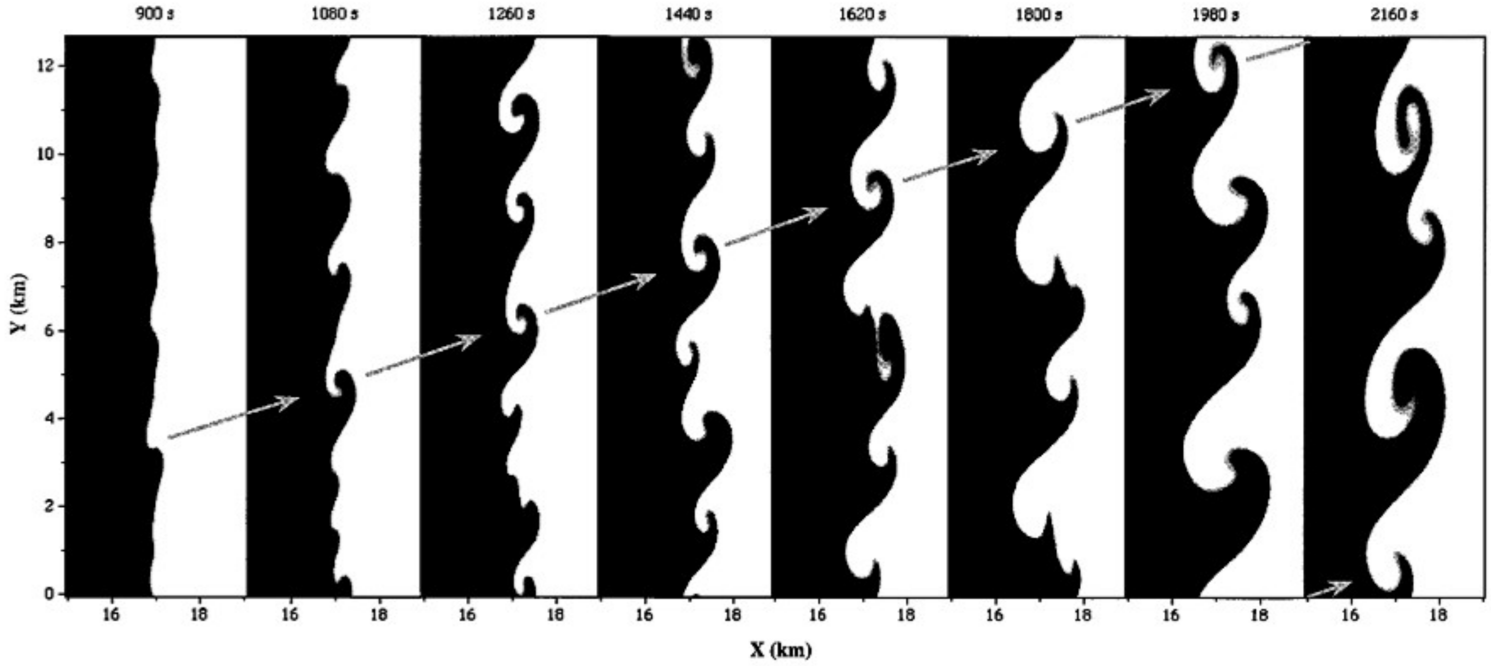
- Straightforward coding would look like …
  - Forward step:
    - u2 = u1 + Δt • *[forcing terms]*
  - Centered step:
    - u3 = u1 + 2Δt • *[forcing terms]*
  - Writing all that code out twice is annoying.
- Instead, we will do …
  - For the first time step, *tstep=Δt;* otherwise, *tstep=2Δt*
  - And so our equations look like …
    - u3 = u1 + tstep • *[forcing terms] (same for W, P)*
    - works because we *also* initialize our arrays u1=u2=0 *(same for W, P)*
  - *Except* for the temperature: θ is *always* forward in time.

Program 5: Where do I start?

- Suggested order of development for Program 5
  - Modify program 2 or 3 code to everywhere to assume NX ≠ NZ
  - Change physical dimensions; domain no longer [-0.5 :+0.5]
  - Create initial θ field, plot $\theta - \bar{\theta}$; verify it looks OK.
    - So the initial θ plot will look like a circular field surround by zeroes
  - Set up all arrays:
    - u, w, p: three time levels, 2-D
    - t: two time levels, 2-D
    - density (for p/t/u levels) and density for w-levels: 1-D
  - Create initial 1-D base-state fields for density
  - Create routine for boundary conditions
  - Test in this order: *PGF*; *linear* θ *advection* θ; θ *diffusion*.
  - *Now*: full physics.

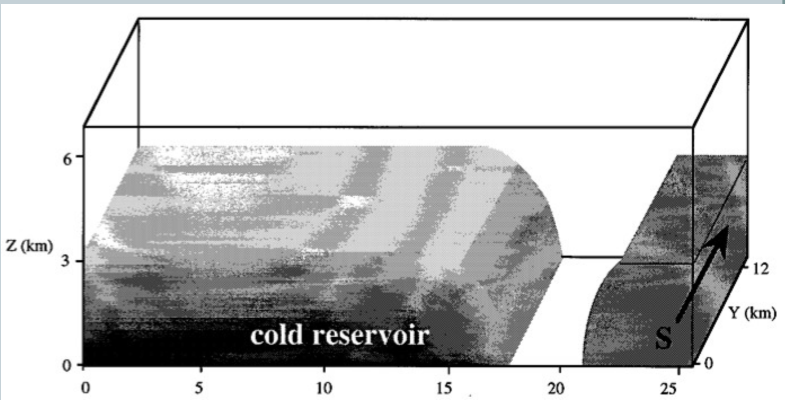
Modeling 3d density currents

NONLINEAR COMPRESSIBLE FLOW



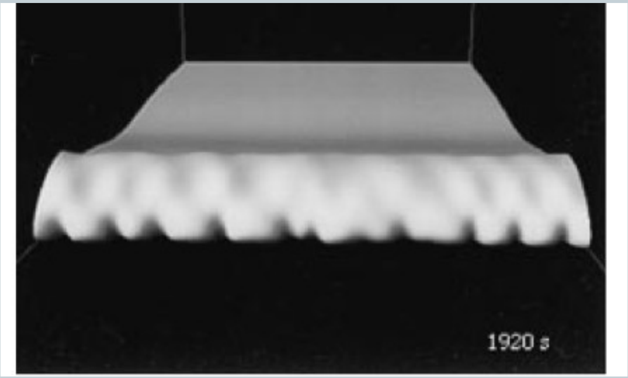
Lee and Wilhelmson (1997)

- Much in common w/computer problem #6 --
- Related to high plains 'landspout' tornadoes
- Configuration:
  - 3-D, $\Delta x=\Delta y=\Delta z=100$m, dry, quasi-compressible, C-grid
  - Random sfc θ perturbations @ T=0
  - Semi-slip surface: $\left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)_{sfc}=\frac{-C_D|\vec{V}|}{\Delta z}(u,v)$
  - BC: Open X, periodic Y
  - Density current encounters significantly different ***V***



Lee and Wilhelmson (1997)

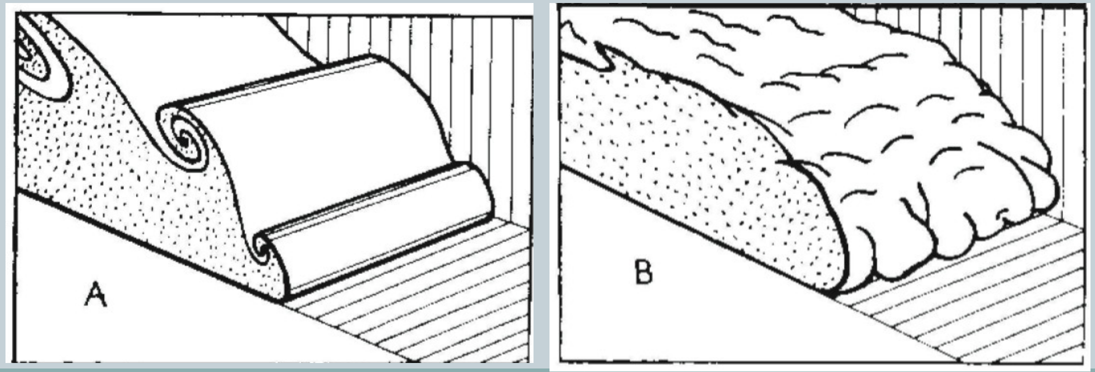
- **Evolution**, step 1:
  - *Random temp. perturbations + semi-slip surface produces* ***lobe and cleft*** *instability along leading edge of density current*



Lee/Wilhelmson Fig. 5: density current leading edge: lobe and cleft instability

Program 6: uses $\Delta U'_{T=0}$

Kelvin-Helmholtz



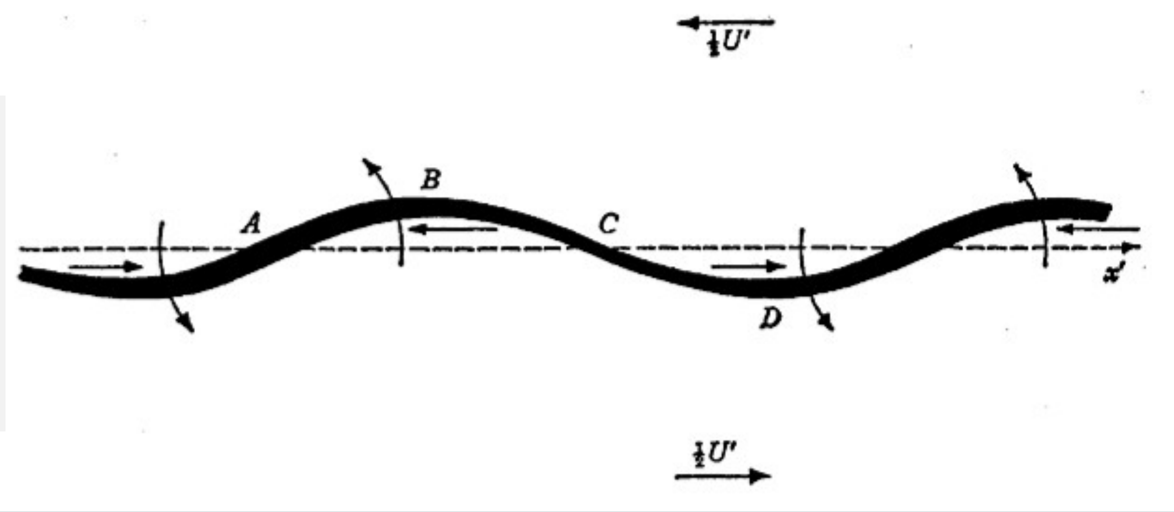
Lobe & Cleft

Simpson (1997)

Lee and Wilhelmson (1997)

- **Evolution**, step 2:
  - *Density current to V>0 region:* ***vortex sheet***
  - *Perturbations on density current lead to* ***horizontal shearing instability*** *(HSI) at leading edge*

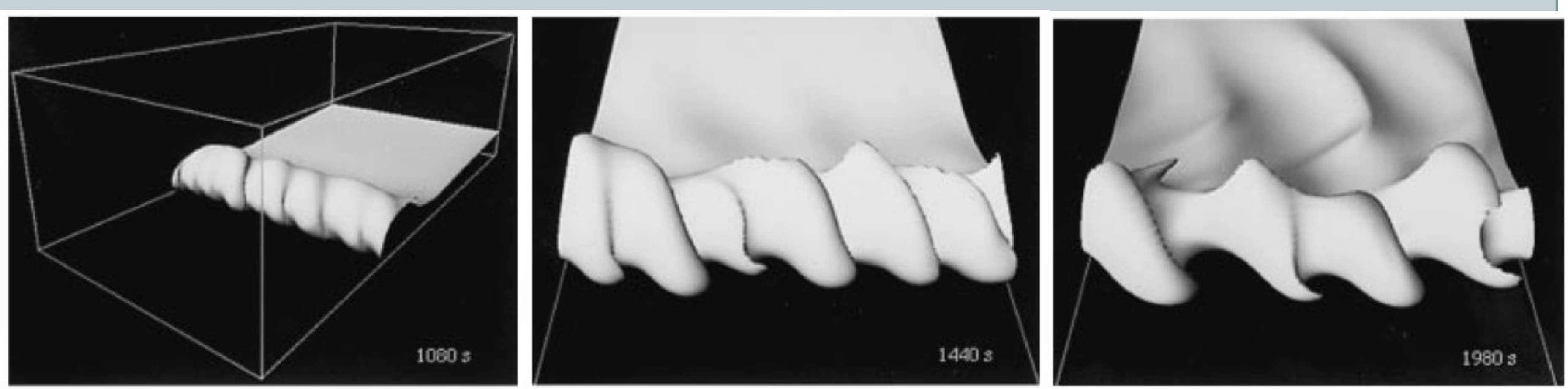
Figure 2: Growth of perturbations on vortex sheet (Batchelor 1967)



Lee and Wilhelmson (1997)

- **Evolution**, step 3:
  - *Circulations with HSI evolve:*
    - *vortex sheet roll up*
    - *subharmonic interaction*
    - *consolidation, dissipation*

Figure 4: Evolution of leading edge of density current.



Transition from wavenumber 8, to 6, to 4

Lee and Wilhelmson (1997)

- **Evolution of leading edge vorticity**

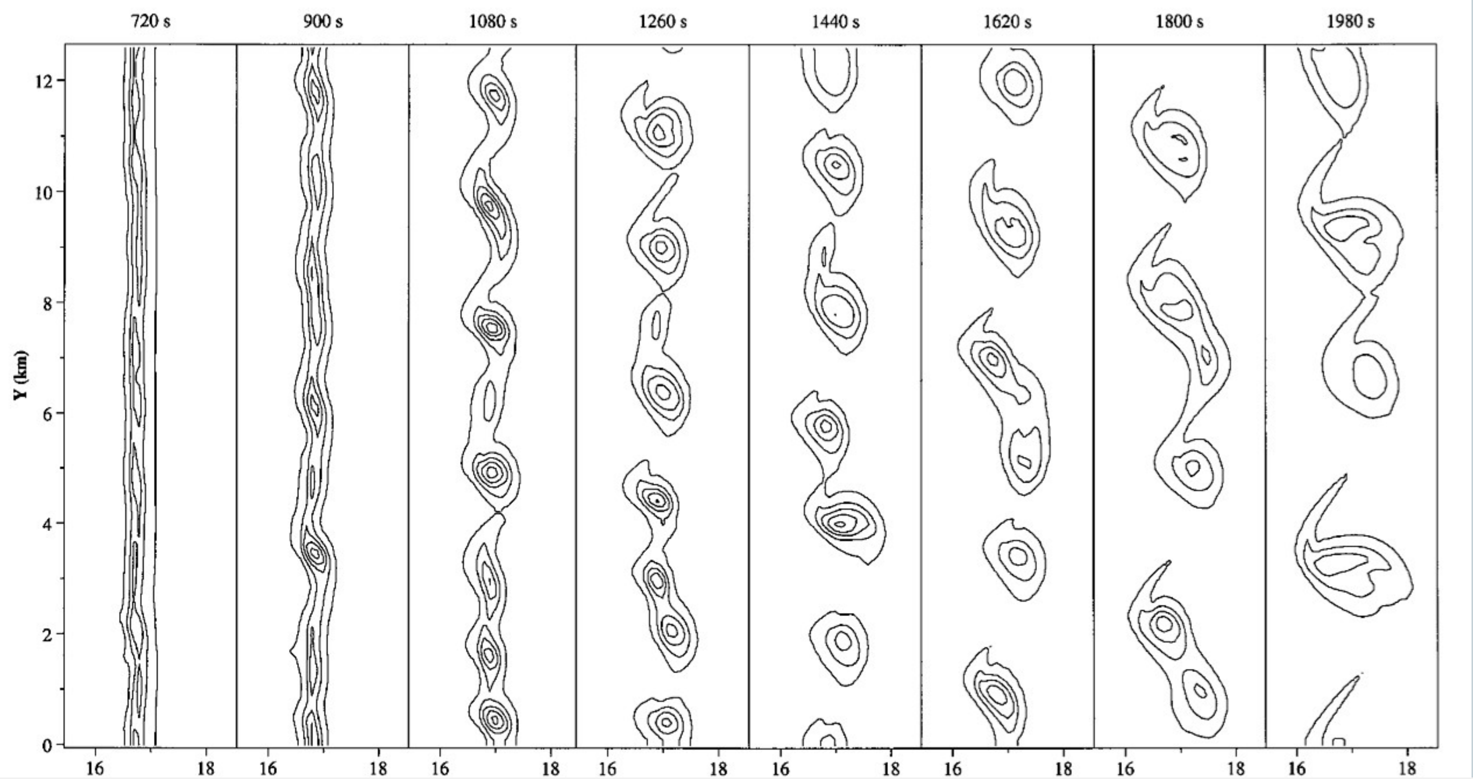


Figure 9: x-y plot of vertical vorticity at leading edge of the density current