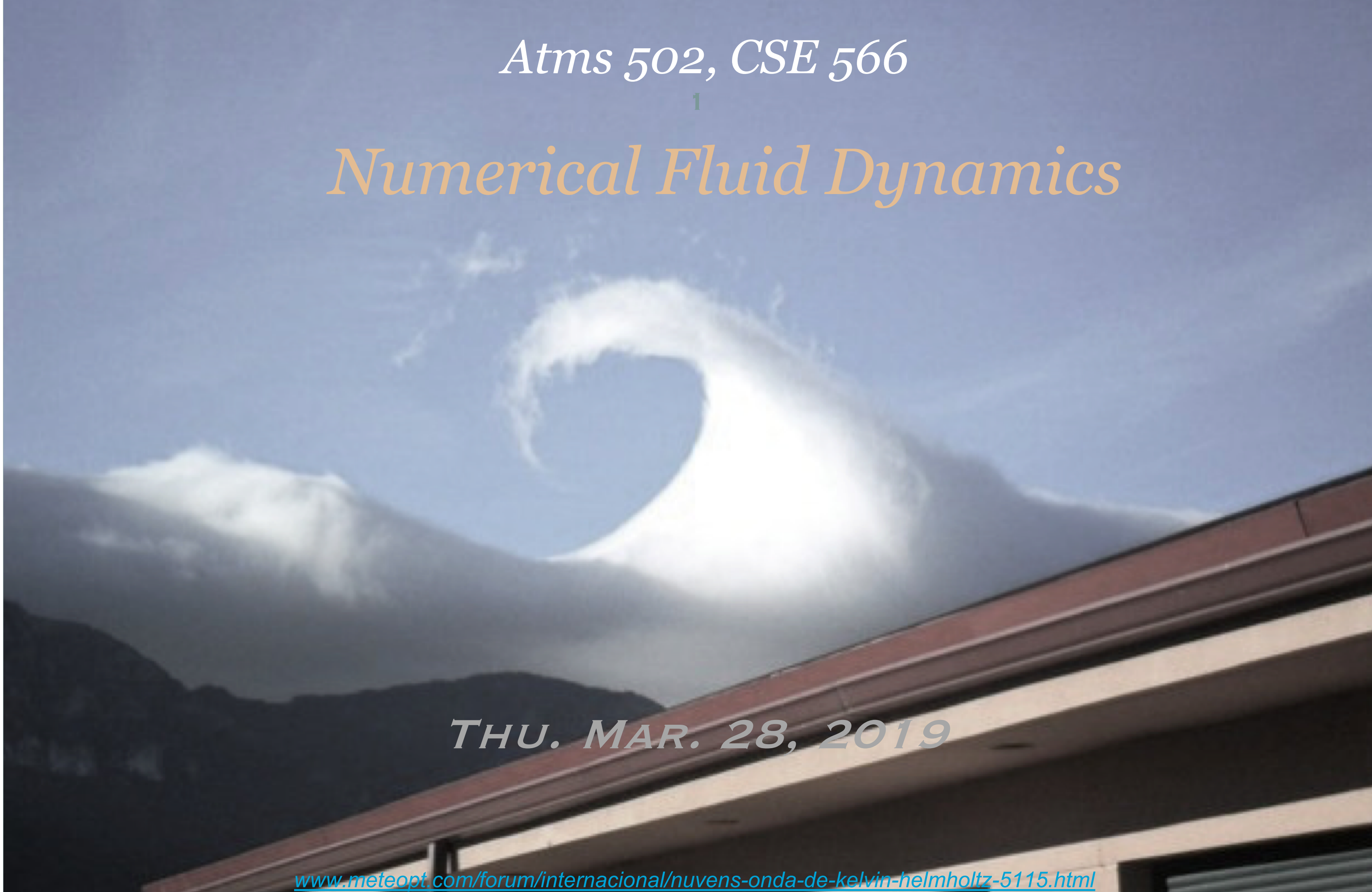
“Beautiful photographs of clouds resulting from the Kelvin-Helmholtz instability photographed in the Valley of Mena (Burgos)”
www.meteopt.com/forum/internacional/nuvens-onda-de-kelvin-helmholtz-5115.html

Atms 502, CSE 566

1

Numerical Fluid Dynamics



Thu. Mar. 28, 2019

www.meteopt.com/forum/internacional/nuvens-onda-de-kelvin-helmholtz-5115.html

ATMS 502
CSE 566

Thursday,
28 March 2019

Class #20

Plan for Today

- 1) Review
  - Time differencing; table
- 2) Nonlinear instability & aliasing
  - Problems, and methods to mitigate them
- 3) Quasi-compressible system
  - On to nonlinear problems & methods
- 4) Semi-Lagrangian methods
  - Intro. to advantages, questions

Aliasing, nonlinear instability, and conservation

References:

Durran section §4.4.1, 4.5

Robert wilhelmson notes

Haltiner and williams section 5-11-1

Paul Schopf notes, school of computational sciences, George mason university (mason.gmu.edu)

Review: Nonlinear doubling

- Inviscid Burger's equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \text{- or -} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

  - The problem for finite differences lies in the multiplication.*

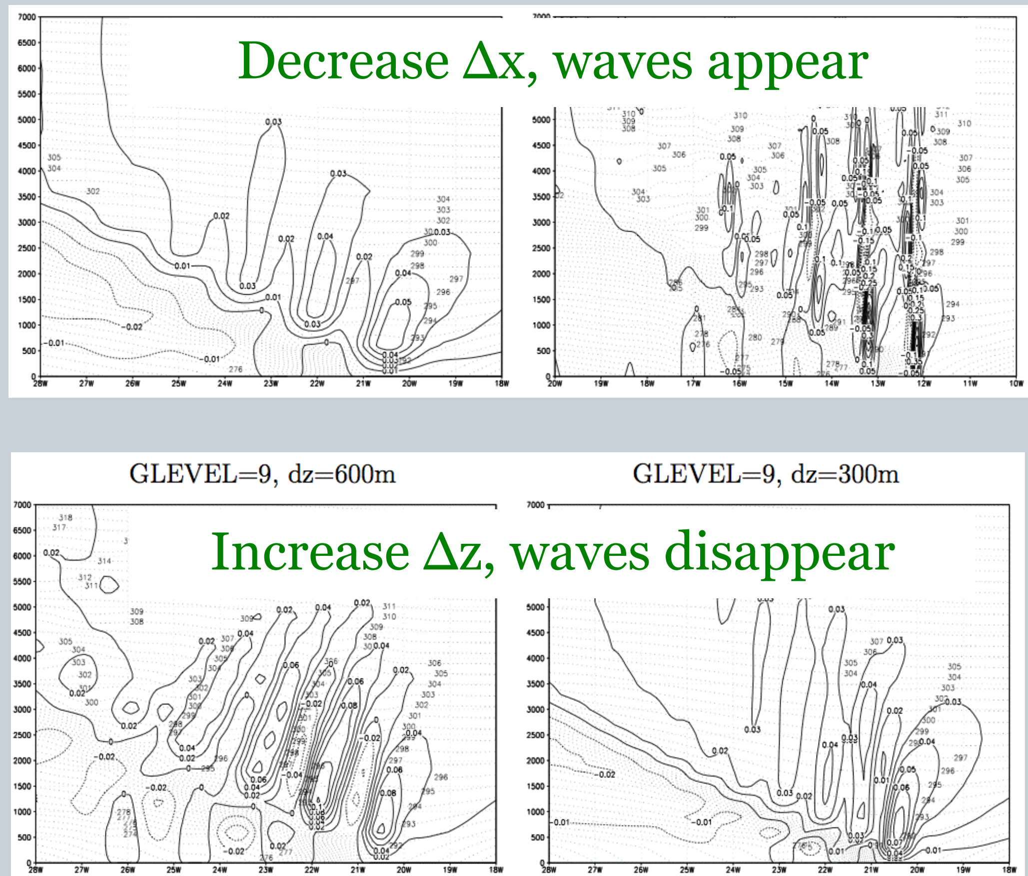
- Let u=sin(kx); then

$$u = \sin kx \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(u^2\right) = k\sin 2kx$$

  - Effective wavenumber is now *2k*.
  - If wavenumber exceeds kmax=(2pi/(2Δx)): *unresolvable*.

Resolution, dimensions and aliasing

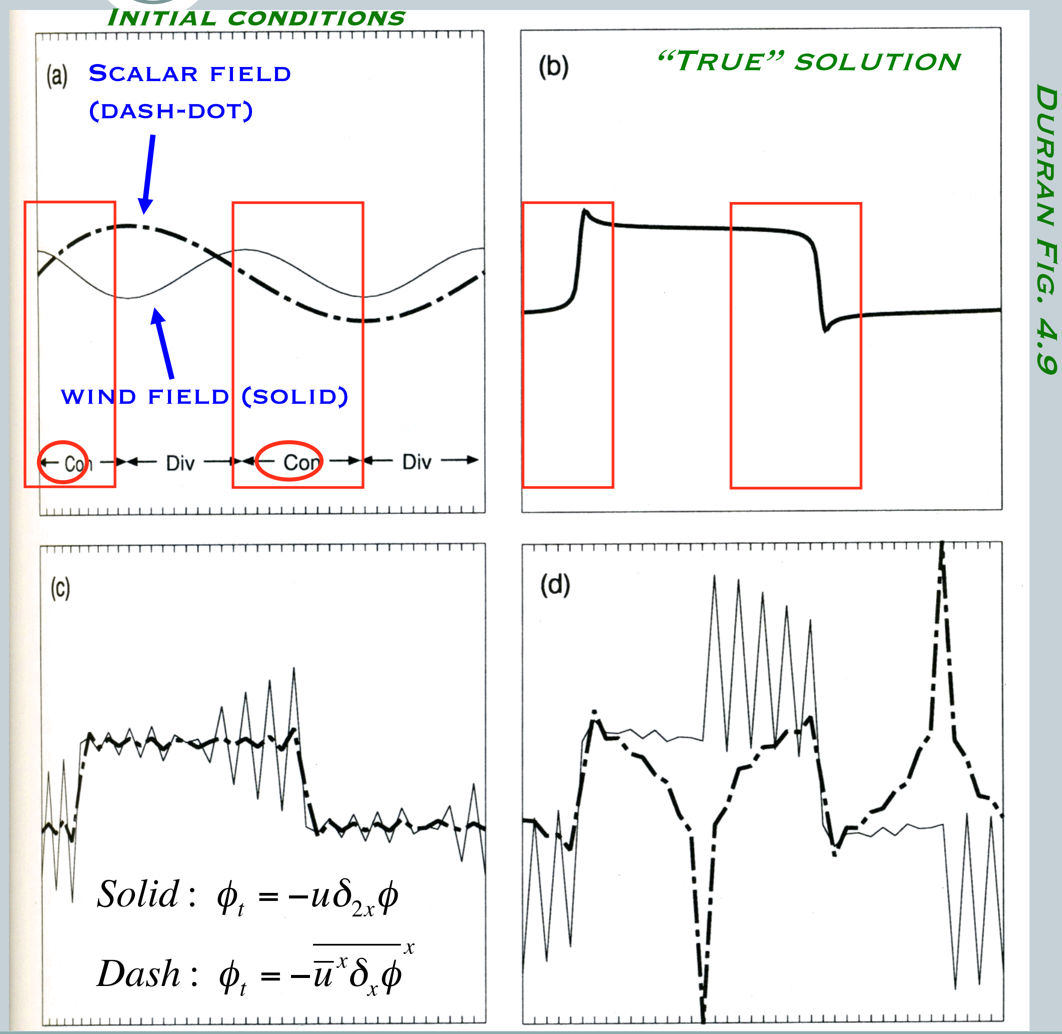
- In some flows, there can be a required link between horizontal & vertical grid spacing.
  - in this *particular* case, $\Delta z/\Delta x$ must be $\leq 1/100$
  - Δx "too small" or Δz "too large" => spurious waves
    - could be **aliasing**: small *but resolved* wavelengths in one dimension … produce *unresolvable* wavelengths in another.
    - some modelers add ***diffusion*** to remove noise – which lowers the *effective* resolution!!



- Iga (2005)

Sharp gradients from smooth IC

- Simple differential (t) - difference (x) equation results
- Problem: linear equation, variable coefficients
  - in other words, c=c(x), but not (t)
- Small scales grow preferentially

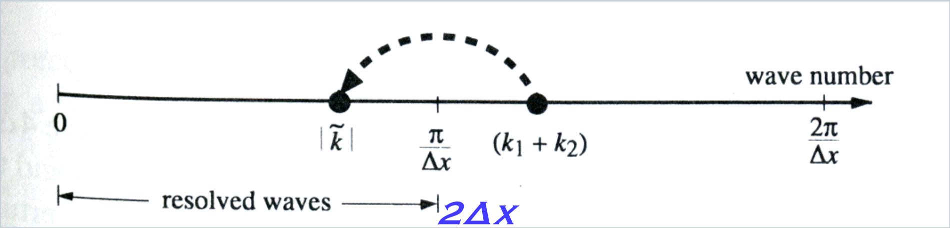


Aliasing

- Where do unresolved waves "go" ?
  - *Durran eqn. 3.91:*

$$\tilde{k} = \begin{cases} k_1 + k_2 - \dfrac{2\pi}{\Delta x}, & \text{if } k_1 + k_2 > \dfrac{\pi}{\Delta x} \\ k_1 + k_2 + \dfrac{2\pi}{\Delta x}, & \text{if } k_1 + k_2 < \dfrac{-\pi}{\Delta x} \end{cases}$$

  - *Examples:*
    - 2Δx • 2.5Δx ⟶ 10Δx
    - (4/3)Δx ⟶ 4Δx
  - Note if <u>*both*</u> waves are 4Δx or longer: ***no aliasing***

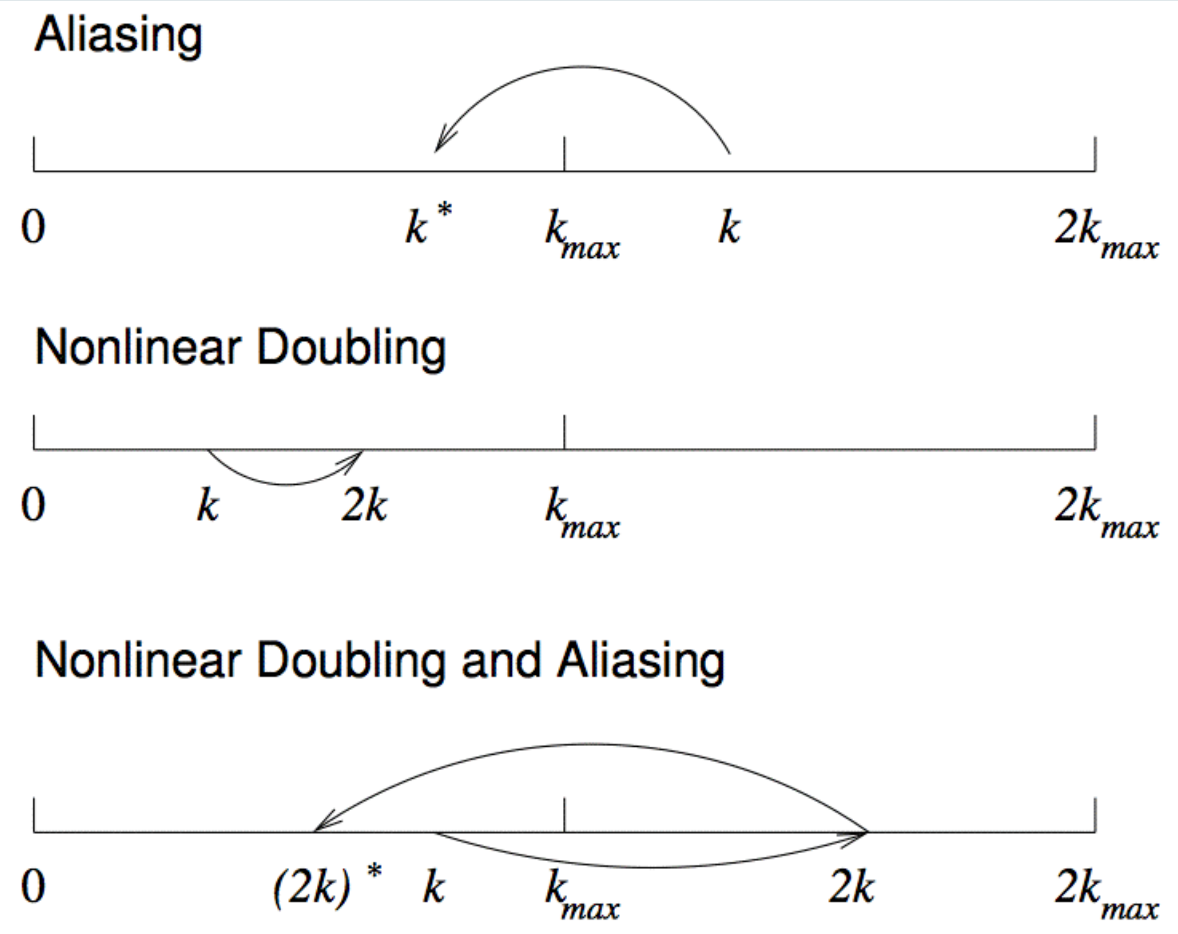


LARGER *WAVENUMBER* ⟶

⟵ LONGER *WAVELENGTHS*

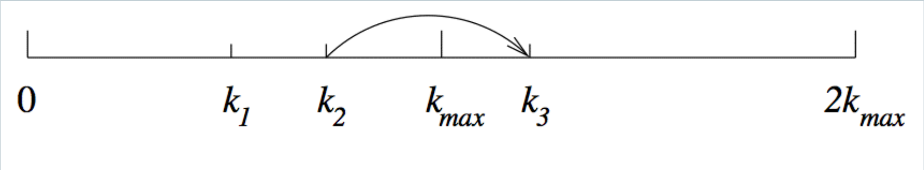
Aliasing

- Schopf (2005) - *Nonlinear doubling* and *aliasing*
- Energy "folded" into low *k*
- Short waves generate long wave energy



Nonlinear instability

- If energy flows into wavenumbers *just above* k_{max} ...

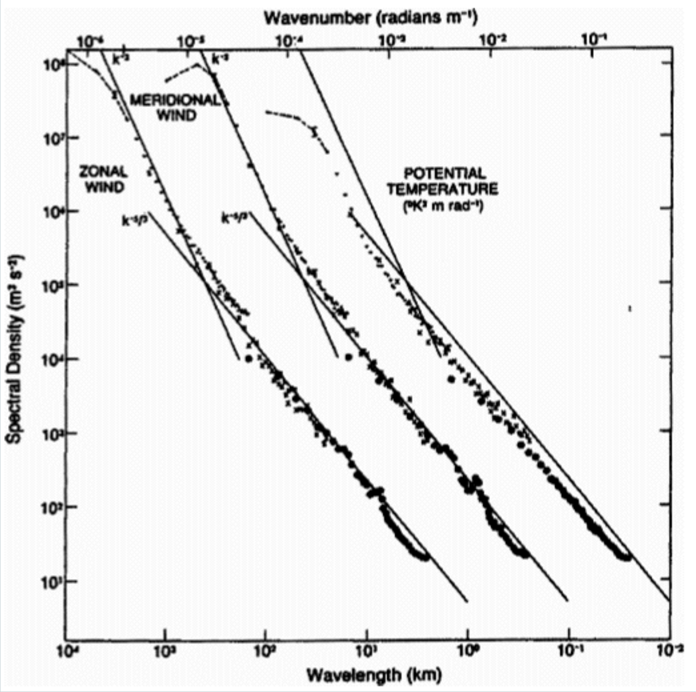


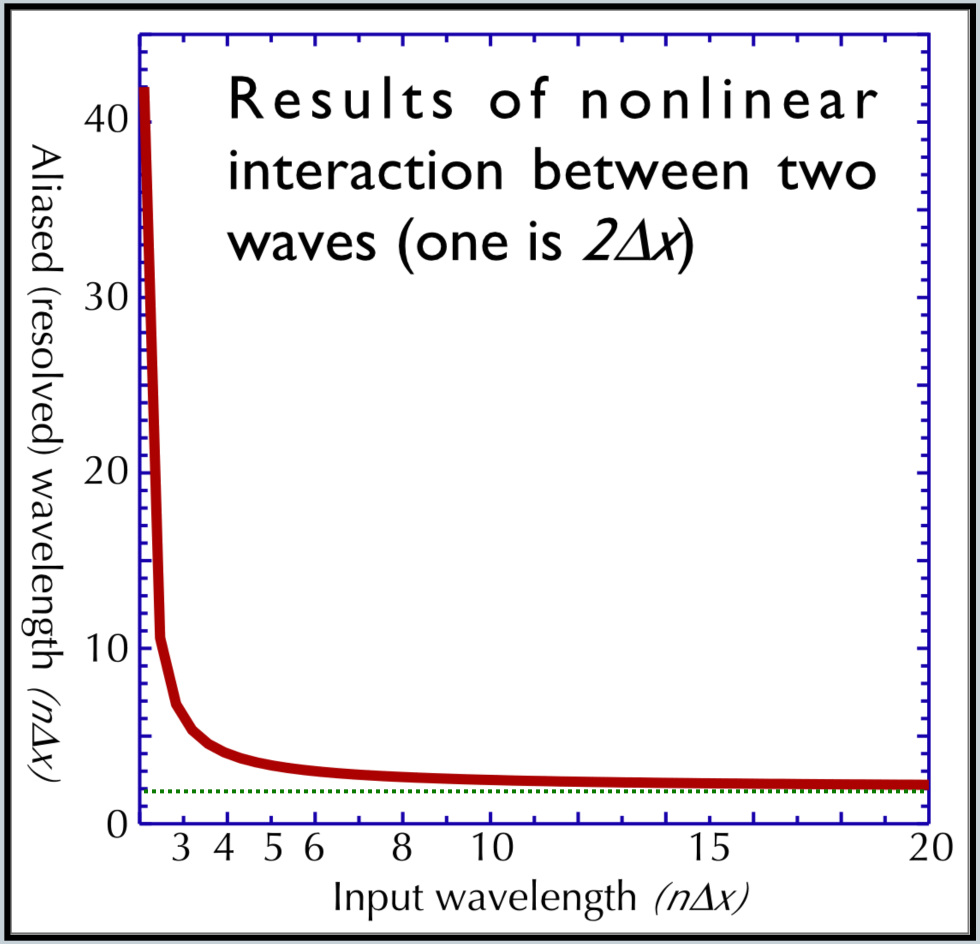
- Aliasing folds it back to wavenumbers *just below* k_{max}
  - builds up energy near grid resolution limit
  - Further nonlinear interaction enhances flow into wavenumbers just above k_{max}: accelerates process
  - This is nonlinear instability.
- What about amplitude?
  - More energy at low *k*. Say k_2 just below kmax ...
  - Nonlinear (k_1+k_2) has more energy *if* k_1 *small*.

www.scs.gmu.edu/climate/courses/csi756/

Nonlinear instability

- $2\Delta x \cdot 2.5\Delta x \longrightarrow 10\Delta x$
- $2\Delta x \cdot 8\Delta x \longrightarrow 2.7\Delta x$
- *more energy: small k*

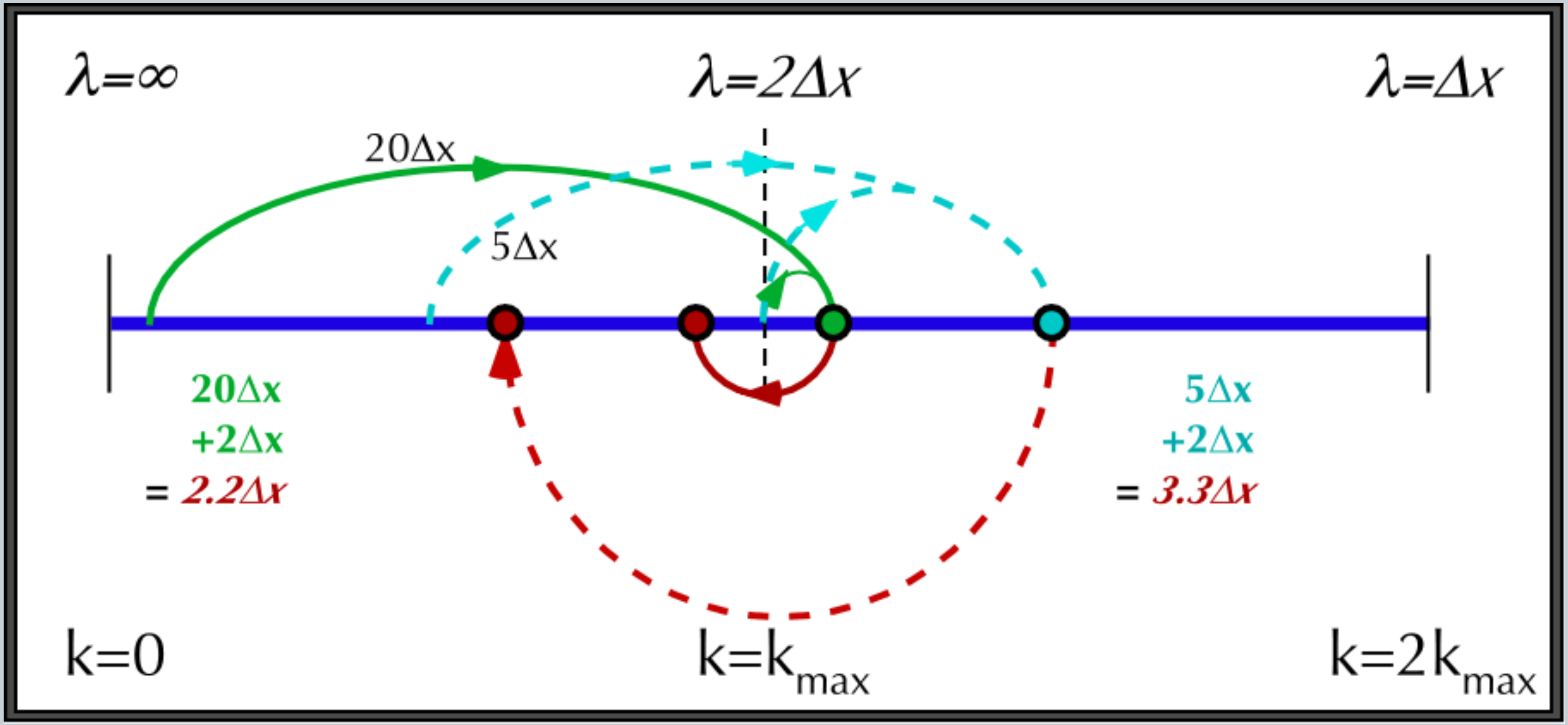




Aliasing in wavenumber space

Input *wave*	Input $k / kmax$	+ $2\Delta x$ = $k / kmax$	aliased $k / kmax$	= final $N\Delta x$
2.1 Δx	0.95	1.95	0.05	42 Δx
5.0 Δx	0.40	1.40	0.60	3.3 Δx
10 Δx	0.20	1.20	0.80	2.5 Δx
20 Δx	0.10	1.10	0.90	2.2 Δx

Aliasing in wavenumber space



Nonlinear instability

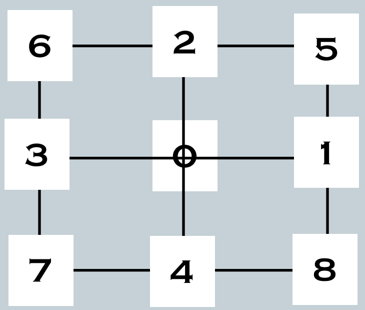
- What to do?
  - Apply spectral filter to remove energy from wavenumbers > $k_{max}/2$
  - Apply Lax-Wendroff differencing -
    - damps high wavenumbers
  - Changed form of finite differencing -
    - Arakawa (1972)
    - Durran example
  - Other methods (e.g. monotonicity, flux limiting)

Nonlinear instability

- Arakawa Jacobian*

$$\frac{\partial \zeta}{\partial t} + v \cdot \nabla \zeta = 0 \quad \text{and} \quad u = -\frac{\partial \psi}{\partial y},\; v = \frac{\partial \psi}{\partial x},\; \zeta = \nabla^2 \psi$$

$$\Rightarrow \quad \frac{\partial \zeta}{\partial t} = -J(\psi,\zeta), \text{ where } J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

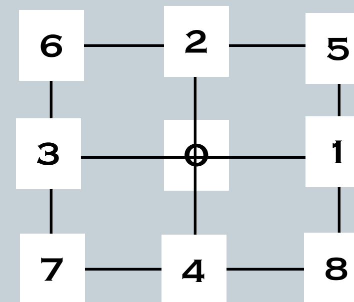


  - sought form of Jacobian ***J*** that conserved
    - average *kinetic energy*
    - average *enstrophy*

*following Haltiner & Williams

Nonlinear instability

- Arakawa Jacobian
  - ($J^{++} + J^{+x} + J^{x+}$)/3 conserves KE, enstrophy, and *mean wavenumber*
  - Prevents nonlinear instability
    - but: *time differencing; still need smoothing*



$$Jacobian = \begin{cases} J^{++} = \left[(\psi_1 - \psi_3)(\zeta_2 - \zeta_4) - (\psi_2 - \psi_4)(\zeta_1 - \zeta_3)\right]\dfrac{1}{4\Delta x^2} \\ J^{+x} = \left[\psi_1(\zeta_5 - \zeta_8) - \psi_3(\zeta_6 - \zeta_7) - \psi_2(\zeta_5 - \zeta_6) + \psi_4(\zeta_8 - \zeta_7)\right]\dfrac{1}{4\Delta x^2} \\ J^{x+} = \left[\zeta_1(\psi_5 - \psi_6) - \zeta_4(\psi_8 - \psi_7) - \zeta_1(\psi_5 - \psi_8) + \zeta_3(\psi_6 - \psi_7)\right]\dfrac{1}{4\Delta x^2} \end{cases}$$

Nonlinear instability

- Burger's Equation – Durran § 4.5
  - Durran shows differential-difference solution to (nonlinear, undamped) Burger's equation
  - True solution: shock forms.
  - Numerical solution:
    - advective form: $$\frac{d\phi_j}{dt} + \phi_j\left(\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}\right) = 0$$
    - flux form: $$\frac{d\phi_j}{dt} + \frac{1}{2}\left(\frac{\phi_{j+1}^2 - \phi_{j-1}^2}{2\Delta x}\right) = 0$$
    - "conservative" form: $$\frac{d\phi_j}{dt} + \frac{\phi_j}{3}\left(\frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x}\right) + \frac{1}{3}\left(\frac{\phi_{j+1}^2 - \phi_{j-1}^2}{2\Delta x}\right) = 0$$

Nonlinear instability

- Burger's Equation - Durran
  - Instabilities in this case appear associated with the formation of shock, but…
  - Shock not required for nonlinear instability
  - Nonlinear instability can develop in smooth flow
    - Example: variable flow (with sinusoidal c(x))
    - Example: viscous Burger's equation
      - there are values of ν for which no shock forms (true case) but for which instability occurs (numerical solution)

$$\frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x} = \nu \frac{\partial^2 \phi}{\partial x^2}$$

The quasi-compressible approximation

Why use this system ?

- Computationally efficient - no elliptic system to solve
- Retains full accuracy *and simple boundary conditions* of the fully compressible system *(Liu et al., JAS, 2003)*
- 3-4x speed of fully compressible model *(Droegemeier et al.)*

Reference information

- A031 – Boussinesq approximation
- A033 – Quasi-compressible approximation (theory)
- C065 – Quasi-compressible approximation (application)

Quasi-compressible system

(Orf et al., JAS 1996)

$$\frac{\partial u}{dt} = -\vec{V}\cdot\vec{\nabla}u - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{dt} = -\vec{V}\cdot\vec{\nabla}v - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial w}{dt} = -\vec{V}\cdot\vec{\nabla}w - \frac{1}{\bar{\rho}}\frac{\partial p}{\partial z} + g\frac{\theta}{\bar{\theta}} + \nu\nabla^2 w$$

$$\frac{\partial \theta}{\partial t} = -\vec{V}\cdot\vec{\nabla}\theta + Q(x,y,z,t) + \nu\nabla^2\theta$$

$$\frac{\partial p}{\partial t} = -c_s^2\left[\bar{\rho}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial(\bar{\rho}w)}{\partial z}\right] + \nu\nabla^2 p$$

- Written in various forms; common property is explicit setting of the sound speed c_s.
- Here, in $\boldsymbol{p}$

Quasi-compressible sound speed

- <u>Assessment of Q-C system</u>
  - Droegemeier and Wilhelmson examine ratio between elastic (acoustic) energy *E* and KE:

$$\frac{E}{K} = \left[\rho'^2 / \bar{\rho} c_s^2\right] \div \bar{\rho}\left(v^2 + w^2\right)$$

c_s *(m/s)*	E/K
50	10 %
100	2.5
200	0.6
300	0.3
350	0.2

  - Concluded from this (+figures, next slide) that: *"imposed sound speed should be no less than 2x the speed of fastest-moving physical mode"*

Quasi-compressible system(5)

Droegemeier and Wilhelmson, 1987

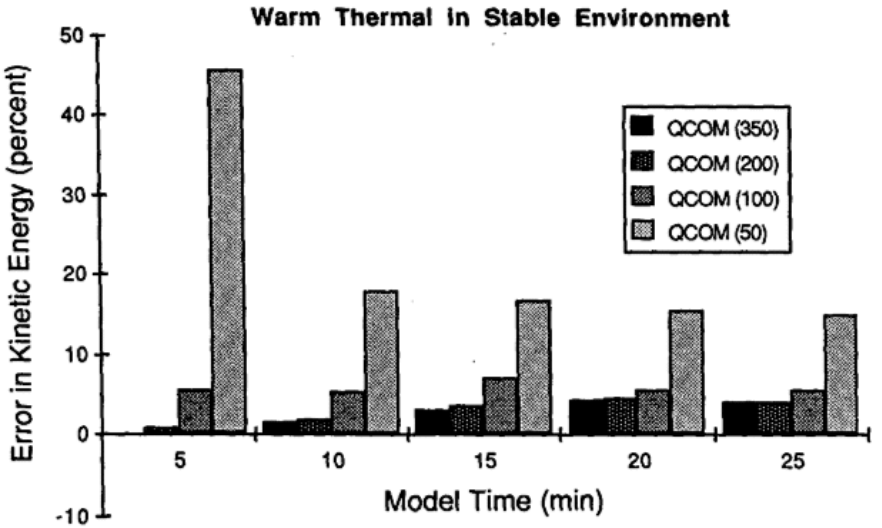


FIG. 4. Error in kinetic energy (percent, relative to the Fully Compressible Outflow Model) for simulations of a warm thermal in a stable environment ($d\bar{\theta}/dz = 4$ K km^{-1}) using several imposed sound wave phase speeds (m s^{-1}, shown in parentheses) in the Quasi-Compressible Outflow Model (QCOM).

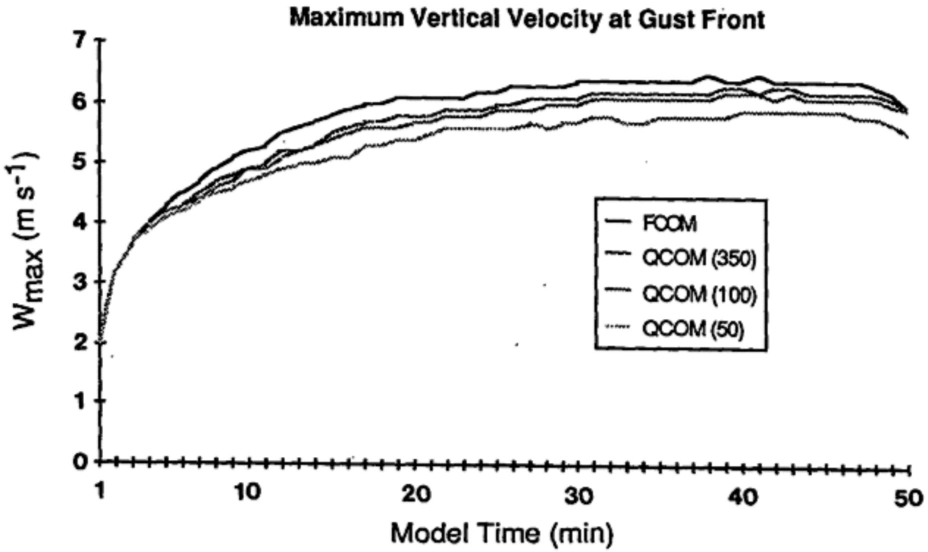


FIG. 5. Time series plots of the maximum vertical velocity induced at the gust front for the fully compressible outflow model (FCOM), and for imposed sound speeds of 350, 100 and 50 m s^{-1} in the quasi-compressible outflow model (QCOM).

Kinetic energy error -

- Small for all but 50 m/s case (grey bars, right)

Maximum vertical velocity

- All appear close (true of most fields)

Semi-Lagrangian methods

Semi-Lagrangian Methods

Ritchie et al. 1995

- "The main motivation for using a semi-Lagrangian formulation is to permit the use of time steps that far exceed the CFL stability criterion for the corresponding Eulerian model … provided that the additional time truncation error does not significantly decrease the accuracy"
- Their case: 4x improvement in efficiency

Semi-Lagrangian Methods

- Generally:
  - Eulerian view - evolution at a point
  - Lagrangian view - following fluid motion
  - *Semi-Lagrangian* viewpoint …
- Semi-Lagrangian methods: find source of tracer arriving at fixed grid locations

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dx}{dt}\frac{\partial F}{\partial x} = 0 \qquad \frac{dx}{dt} = U(x,t)$$