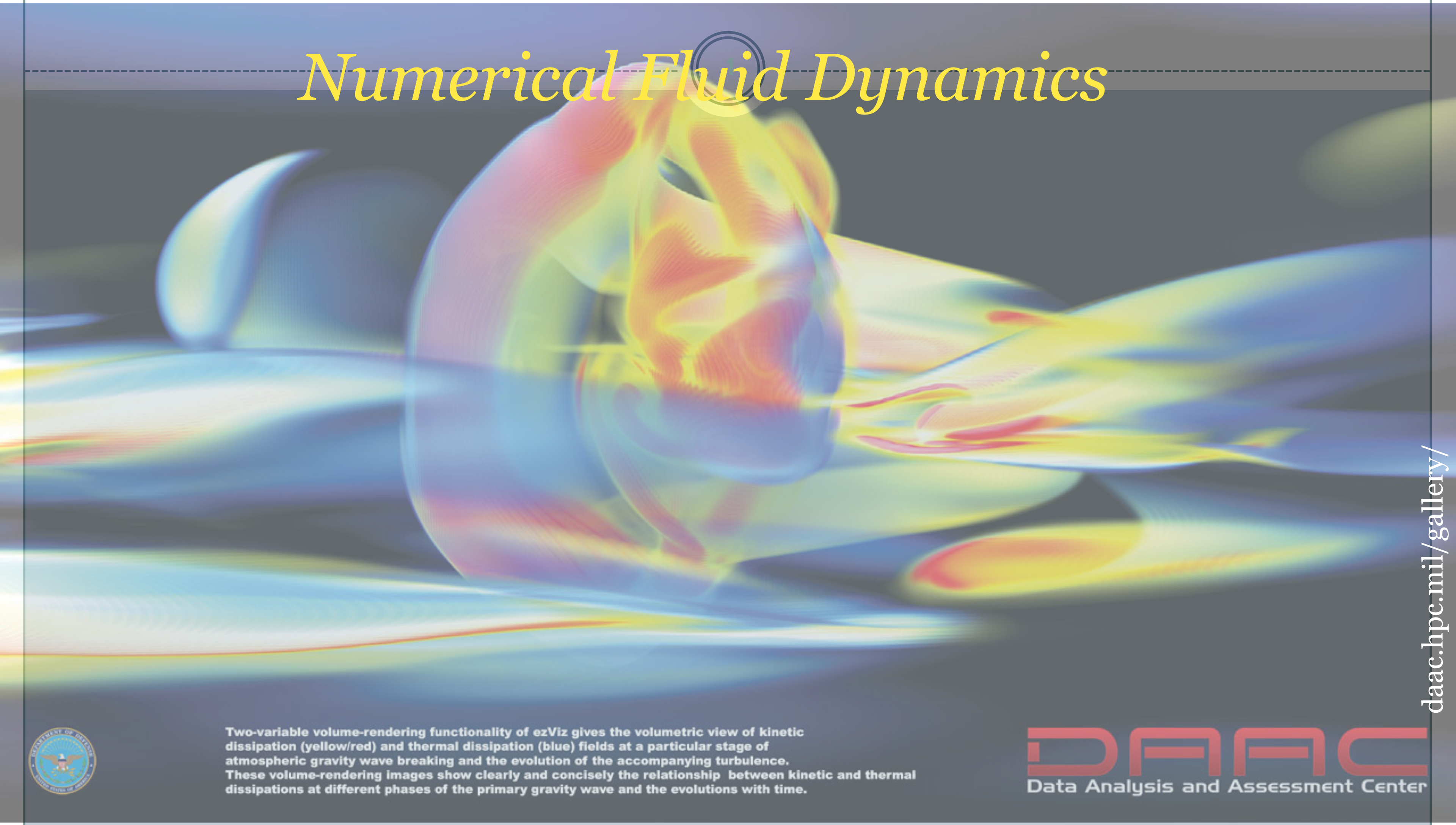
# Atms 502, CSE 566

## Numerical Fluid Dynamics




TUE., MAR. 26, 2019

Image: **Breaking gravity wave & turbulence**

ATMS 502
CSE 566

Tuesday,
26 March 2019

Class #19

# Plan for Today

- 1) Review
  - Visualization
  - HPC architecture
  - Amdahl's law, parallel performance
- 2) Time differencing
  - including time differencing table
- 3) Nonlinear instability & aliasing
  - problems and methods to mitigate them

# Time differencing



## "STAGES" AND "LEVELS"
## HIGHER ORDER ACCURACY

Terminology:

- n, n+1 … time levels
- $\alpha$, $\beta$, … coefficients of spatial derivatives
- *F* approximations to spatial derivatives

# Time differencing: stages vs. levels



- *Review:* time differencing includes
  - 1) how we express the *time derivative*
  - at what *time levels* we evaluate the *spatial derivatives*
- Levels
  - … refers to how many time levels are in our scheme
  - Lax-Wendroff: *2-level*. Leapfrog: *3-level*
- Stages
  - … refers to how many times we evaluate the spatial derivatives
  - Lax-Wendroff: *single-stage*. Runge-Kutta: *2 or more stages*

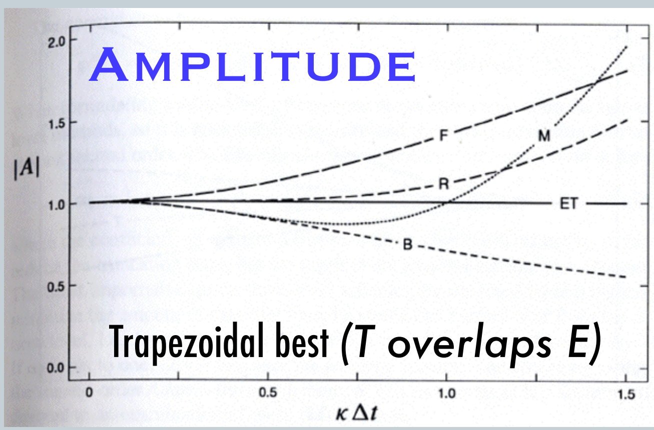
# *Summary:* Single-stage, 2-level schemes

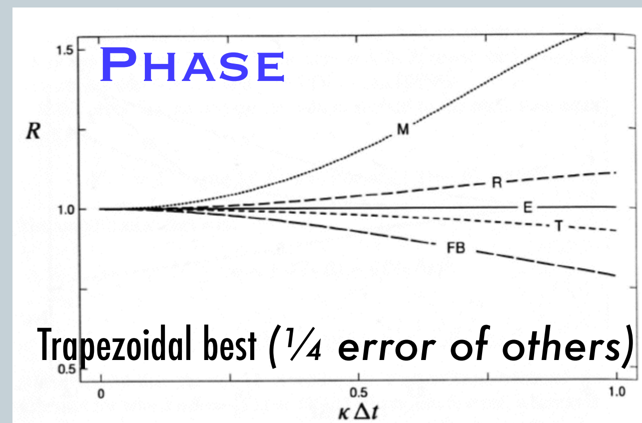


- *Single-stage*: evaluate spatial derivatives <u>once</u>
- 2-level: there are two time levels, $n$ and $n+1$

$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t}=\alpha F\left(\phi^{n}\right)+\beta F\left(\phi^{n+1}\right),\ \alpha+\beta=1$$

<u>Euler method</u>: α=1,β=0
<u>Backward method</u>: α=0,β=1
<u>Trapezoidal method</u>: α=β=1/2




$$|A|^2=1+\left(\alpha^2-\beta^2\right)\left(\frac{\omega^2\Delta t^2}{1+\beta^2\omega^2\Delta t^2}\right)$$

$$R_{forward}=R_{backward}\approx 1-\frac{(\omega\Delta t)^2}{3};\quad R_{trapezoidal}\approx 1-\frac{(\omega\Delta t)^2}{12}$$

Schemes: Forward *F* • Backward *B* • Trapezoidal *T* • 2nd-order Runge-Kutta *R* • Matsuno *M*

# *Summary:* Single-stage, 3-level schemes



- Single-stage: evaluate spatial derivatives once
- 3-level: there are 3 time levels, *n-1, n, n+1*

$$\frac{\phi^{n+1} - \left(\alpha_1 \phi^n + \alpha_2 \phi^{n-1}\right)}{\Delta t} = \beta_1 F\left(\phi^n\right) + \beta_2 F\left(\phi^{n-1}\right)$$

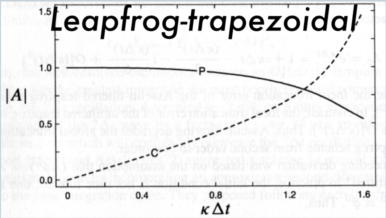
$$\alpha_1 = 1 - \alpha_2; \quad \beta_1 = \frac{\alpha_2 + 3}{2}, \; \beta_2 = \frac{\alpha_2 - 1}{2}$$

($\beta$ restrictions make schemes at least 2nd-order – Durran p. 58)

- Leapfrog: $\alpha_1$=0, $\alpha_2$=1, $\beta_1$=2, $\beta_2$=0 *Time filtering; comp. mode; even/odd ..*
- Leapfrog-trapezoidal *A predictor-corrector method.*

  *2nd-order unlike time-filtered LF; computational mode damped*

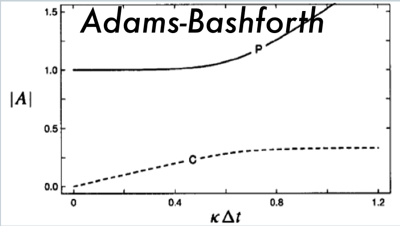
$$\phi^* = \phi^{n-1} + 2\Delta t F\left(\phi^n\right)$$
$$\phi^{n+1} = \phi^n + (\Delta t/2)\left[F(\phi^n) + F(\phi^*)\right]$$




- Adams-Bashforth

  *2nd-order, fwd time, weak instability; computational mode damped. Higher-order versions of A-B exist.*

$$\phi^{n+1} = \phi^n + (\Delta t)\left[\frac{3}{2}F(\phi^n) - \frac{1}{2}F(\phi^{n-1})\right]$$



# *Overview:* Multi-stage (RK)/step methods



- Objective: higher-order accuracy in time
- Multistage: *Durran § 2.3*: evaluate spatial *F* terms at several times between $n\Delta t$, $(n+1)\Delta t$
- Multistep: *Durran § 2.4*: information from prior levels incorporated in integration formula.
  - Multistep: extra storage needed, but fewer evaluations of *F*
  - Multistage *or* multistep: computational modes arise.
  - Multiple forms of RK (Runge-Kutta) methods exist, e.g. low-storage form.
- More steps/or/stages: more (computational) work.
  - Good: less restrictive time step. Bad: *more* computational modes.
  - For more information: see Durran § 2.3.2, RK3 and RK4 methods.

# *Overview:* Higher time accuracy in 3-D



- Problem: order of directional operators matters.
- Theory: *Durran §4.3*: compares exact (that from Taylor series) vs. finite difference results
  - Let's call X-operator $F_1$ … and … Y-operator $F_2$ …
  - Then if $F_1F_2 = F_2F_1$, we say the *operators commute*
- Plan: if operators *don't* commute –
  - We can still get higher temporal accuracy …
  - Use *Strang Splitting*. In 3D: *Durran eq. 4.59 p. 172* –

$$\left[F_1(\Delta t/2)\right]\left[F_2(\Delta t/2)\right]\left[F_3(\Delta t)\right]\left[F_2(\Delta t/2)\right]\left[F_1(\Delta t/2)\right]$$

# Time differencing summary: Durran § 2.6



- Durran table 2.1-2.2: summary of methods
  - *Order of scheme*
  - *Storage factor*
    - *Number of full arrays needed*
    - *Not given if implicit; depends on method*
  - *Efficiency factor*
    - *Largest stable step ÷ by # evaluations of **F***
  - *Max s*
    - *largest stable $\omega\Delta t$*

*$h=\Delta t$; $s=\omega\Delta t$ !!*

**Table 2.1** Summary of methods for the solution of ordinary differential equations. The second- and third-order Runge–Kutta methods are low-storage variants; $h = \Delta t$

| Method | Order | Formulae |
|---|---|---|
| Forward | 1 | $\phi_{n+1} = \phi_n + hF(\phi_n)$ |
| Backward | 1 | $\phi_{n+1} = \phi_n + hF(\phi_{n+1})$ |

DURRAN 2ND ED. TABLES 2.1-2.2, PP 83-84

| Method | Storage factor | Efficiency factor | Amplification factor | Phase error | Max $s$ |
|---|---|---|---|---|---|
| Forward | 2 | 0 | $1+\frac{s^2}{2}$ | $1-\frac{s^2}{3}$ | 0 |

# Aliasing, nonlinear instability, and conservation



References:

Durran section §4.4.1, 4.5

Robert Wilhelmson notes

Haltiner and Williams section 5-11-1

Paul Schopf notes, school of computational sciences, George Mason University (mason.gmu.edu)

# (Nonlinear) advection



- Back to the familiar.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \text{- or -} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

  - The problem for finite differences lies in the multiplication.*

- Let u=sin(kx); then

$$u = \sin kx \quad \Rightarrow \quad \frac{\partial}{\partial x}\left(u^2\right) = k \sin 2kx$$

  - What effective wavenumber are we working with now?

# Aliasing



- Even if u=sin*kx* is resolved…
  - If … (kmax/2) < k < kmax,
  - the nonlinear term puts energy into:

    kmax < k < 2kmax

  - This is not resolvable!
    - ✓ It will appear in other, resolvable wavelengths.
    - ✓ This is aliasing.

# Aliasing: Example



- Some studies have investigated gravity waves near weather fronts.
- They appeared in observations
- They appeared in high-resolution simulations.
- Some of the *modeled* waves were not real.
- They appeared at *high horizontal resolution*.

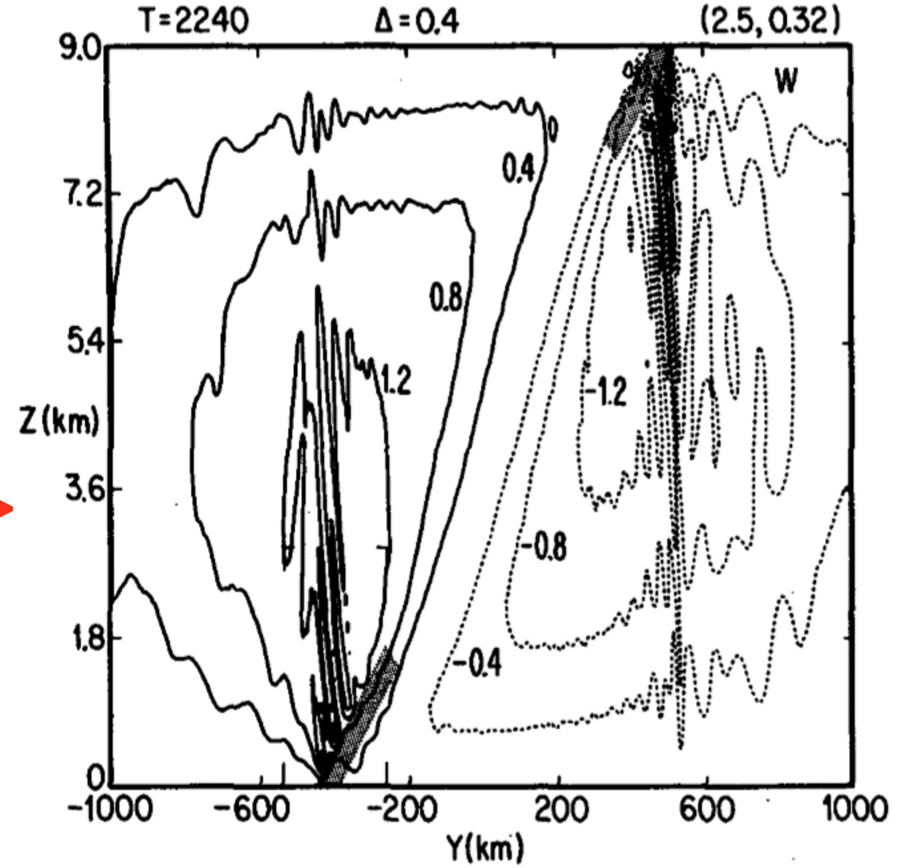
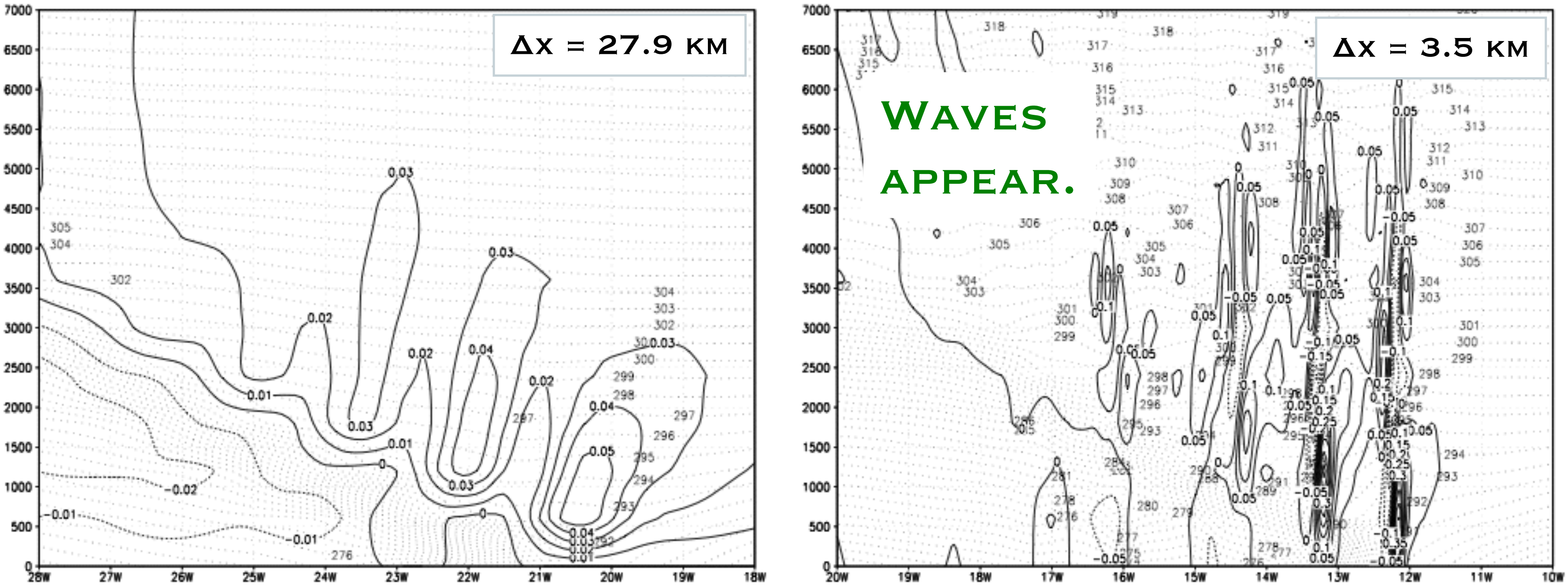



FIG. 2. Vertical velocity at $T$ = 2240 min from the beginning of the experiment. The experiment that produced this figure had a horizontal resolution of 2.5 km and a vertical resolution of 320 m. This

GALL ET AL. 1988

# Spurious waves



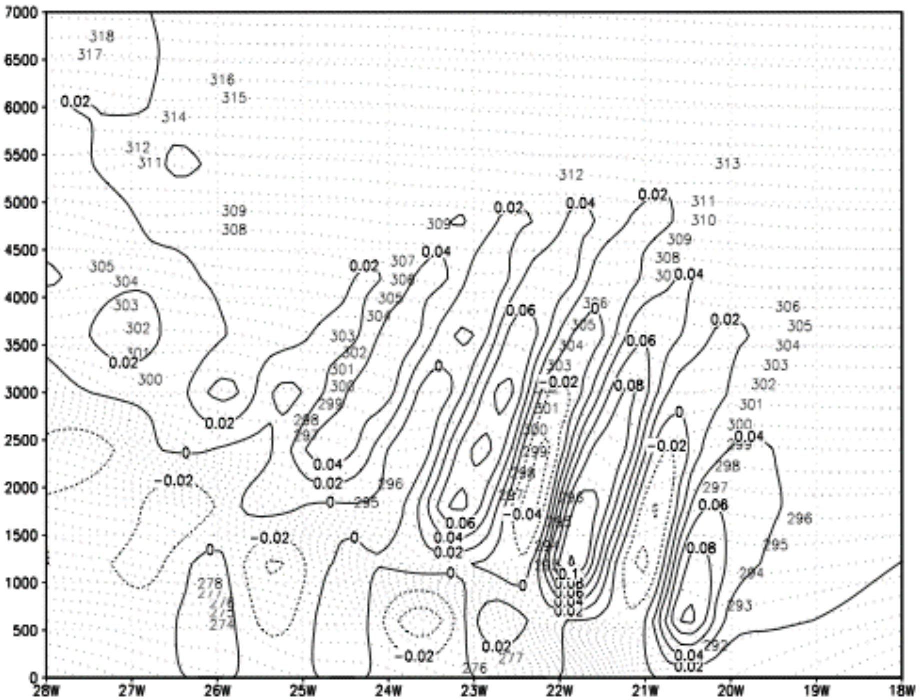



- Iga (2005) - waves on a front. Vertical velocity is shown.
- Only change is decrease in Δx.

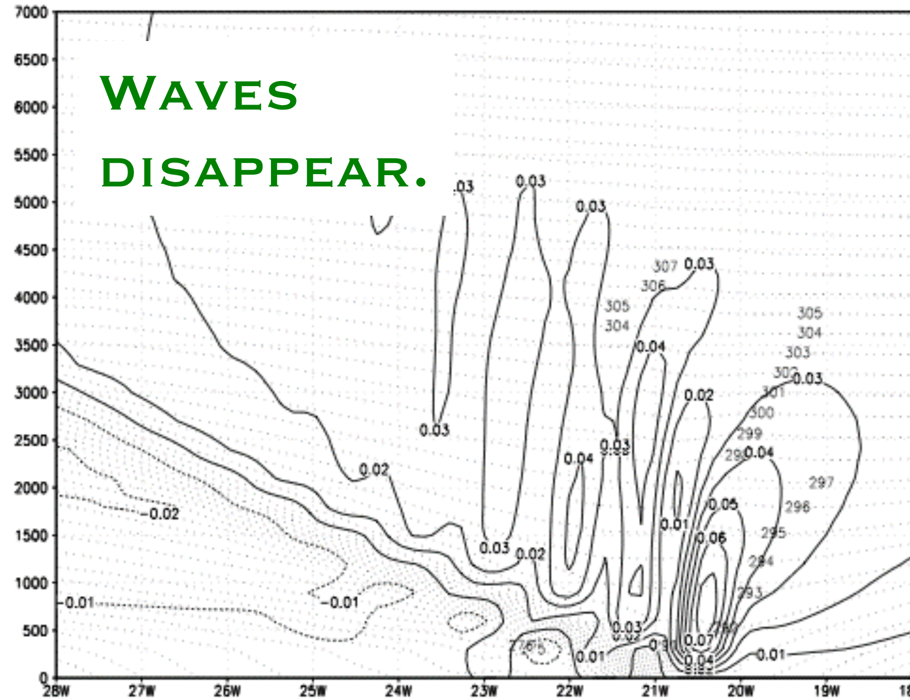
# Spurious waves



GLEVEL=9, dz=600m



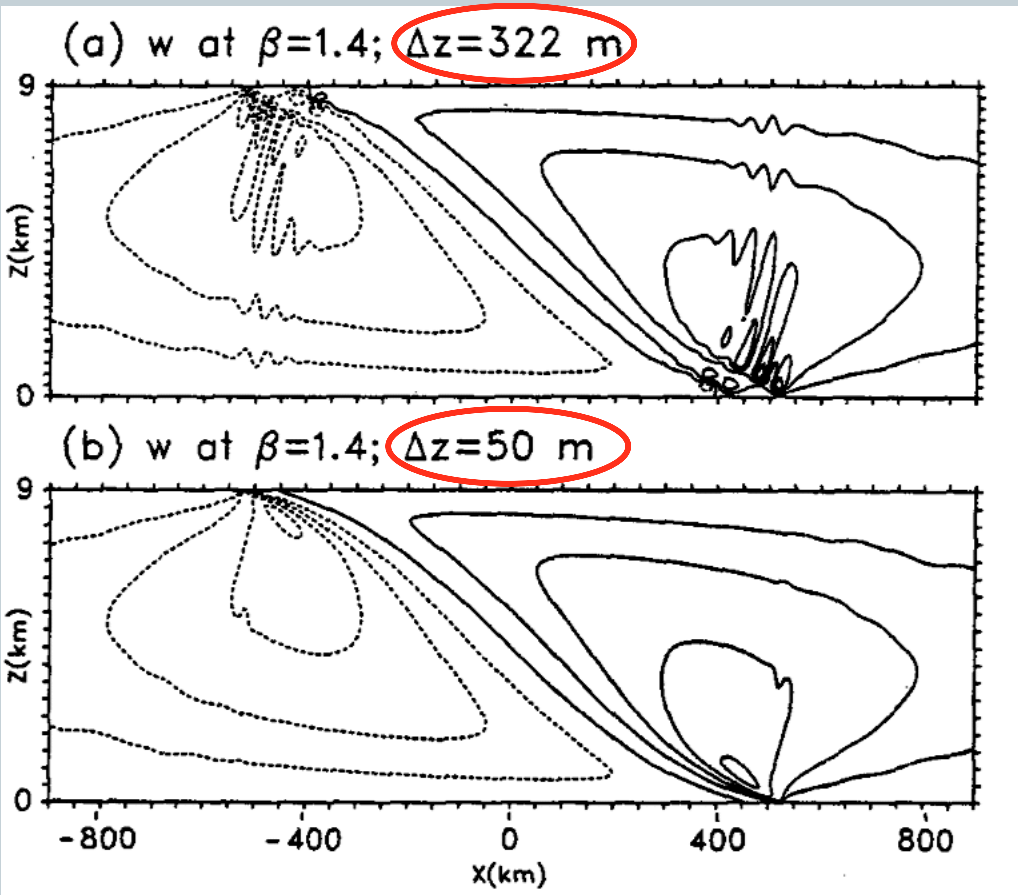
GLEVEL=9, dz=300m




- Iga (2005) - waves on a front. Vertical velocity is shown. Δx=14 km. Only change is decrease in Δz.

# Spurious waves

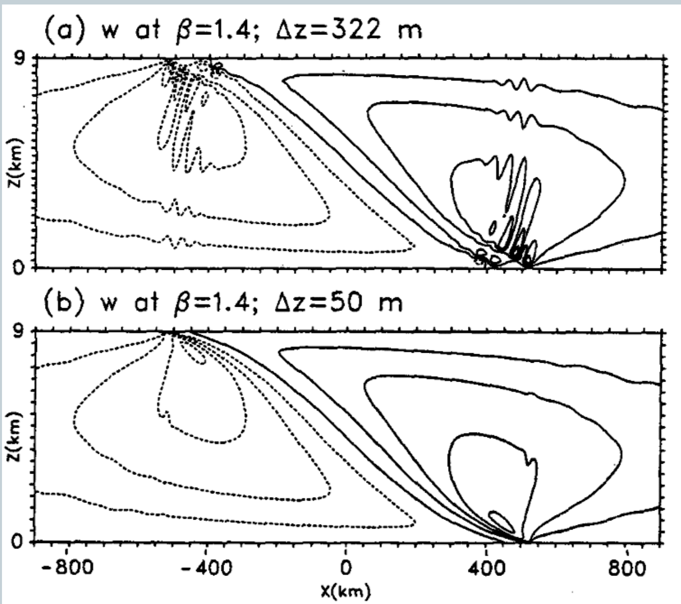


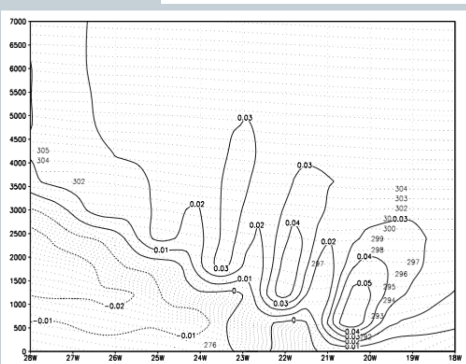


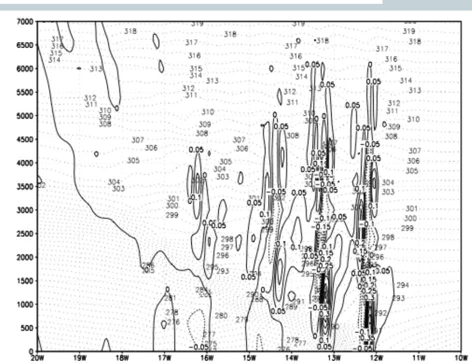

- Snyder et al. 1993. Only change is Δz.
- Waves <u>appear</u> at high *horizontal* resolution
- Waves <u>disappear</u> at increased *vertical* resolution.
- *Why?*

# Spurious waves







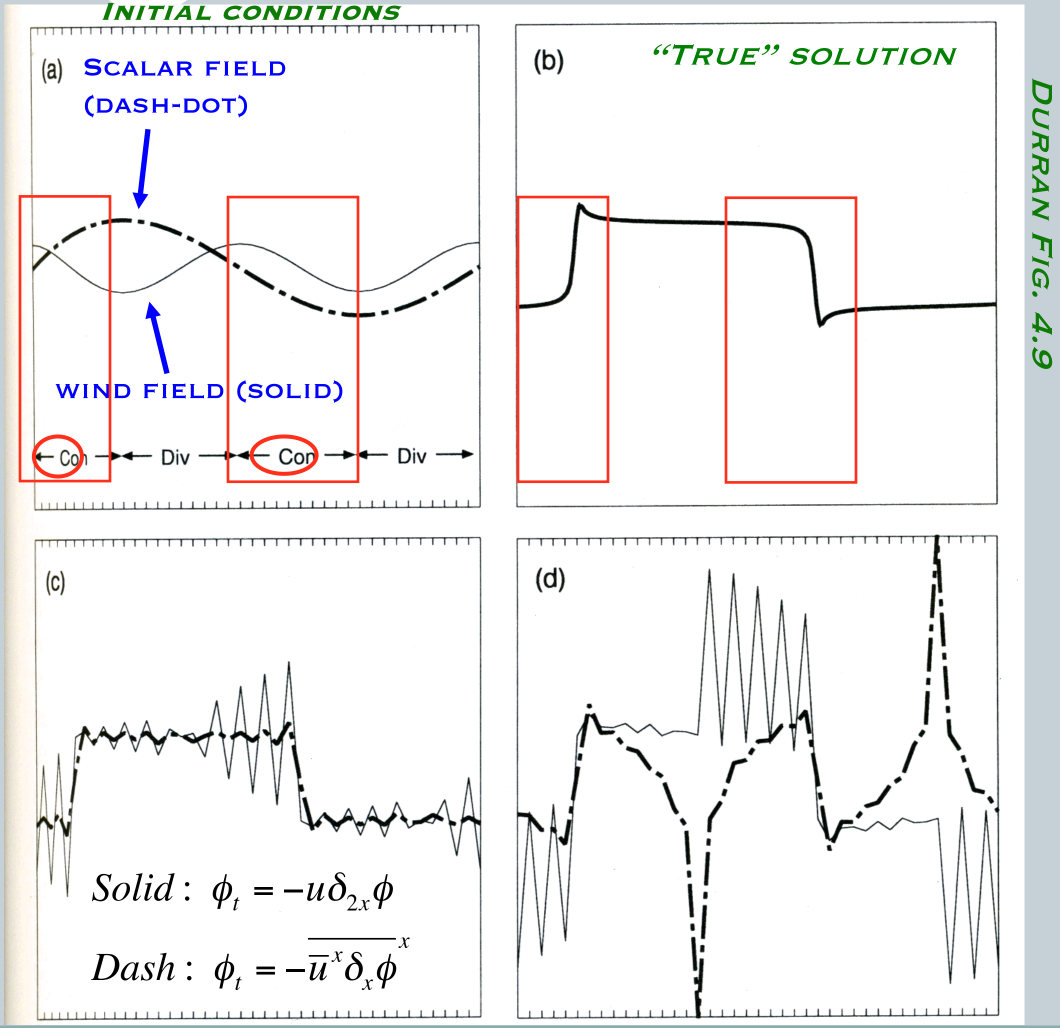
One possibility…

- Reducing $\Delta x$ while keeping $\Delta z$ constant
- .. results in small horizontal waves with small vertical scales … too small to be resolved.
- Uh-oh!

# Aliasing



- Simple differential (t) - difference (x) equation results
- Problem: linear equation, variable coefficients
  - in other words, c=c(x), but not (t)
- Small scales grow preferentially



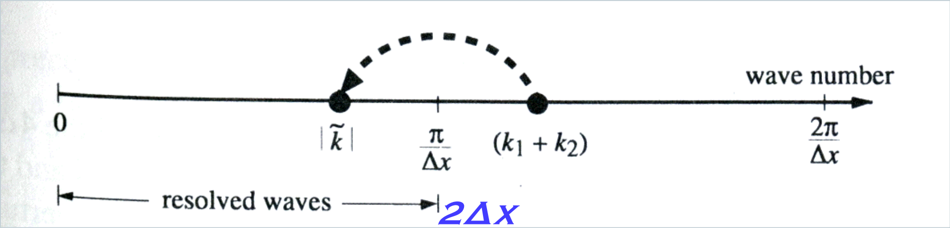
# Aliasing



- Where do unresolved waves "go" ?
  - *Durran eqn. 3.91:*

$$\tilde{k} = \begin{cases} k_1 + k_2 - \dfrac{2\pi}{\Delta x}, \text{ if } k_1 + k_2 > \dfrac{\pi}{\Delta x} \\ k_1 + k_2 + \dfrac{2\pi}{\Delta x}, \text{ if } k_1 + k_2 < \dfrac{-\pi}{\Delta x} \end{cases}$$

  - *Examples:*
    - 2Δx • 2.5Δx ⟶ 10Δx
    - (4/3)Δx ⟶ 4Δx
  - Note if *both* waves are 4Δx or longer: *no aliasing*




LARGER *WAVENUMBER* ⟶

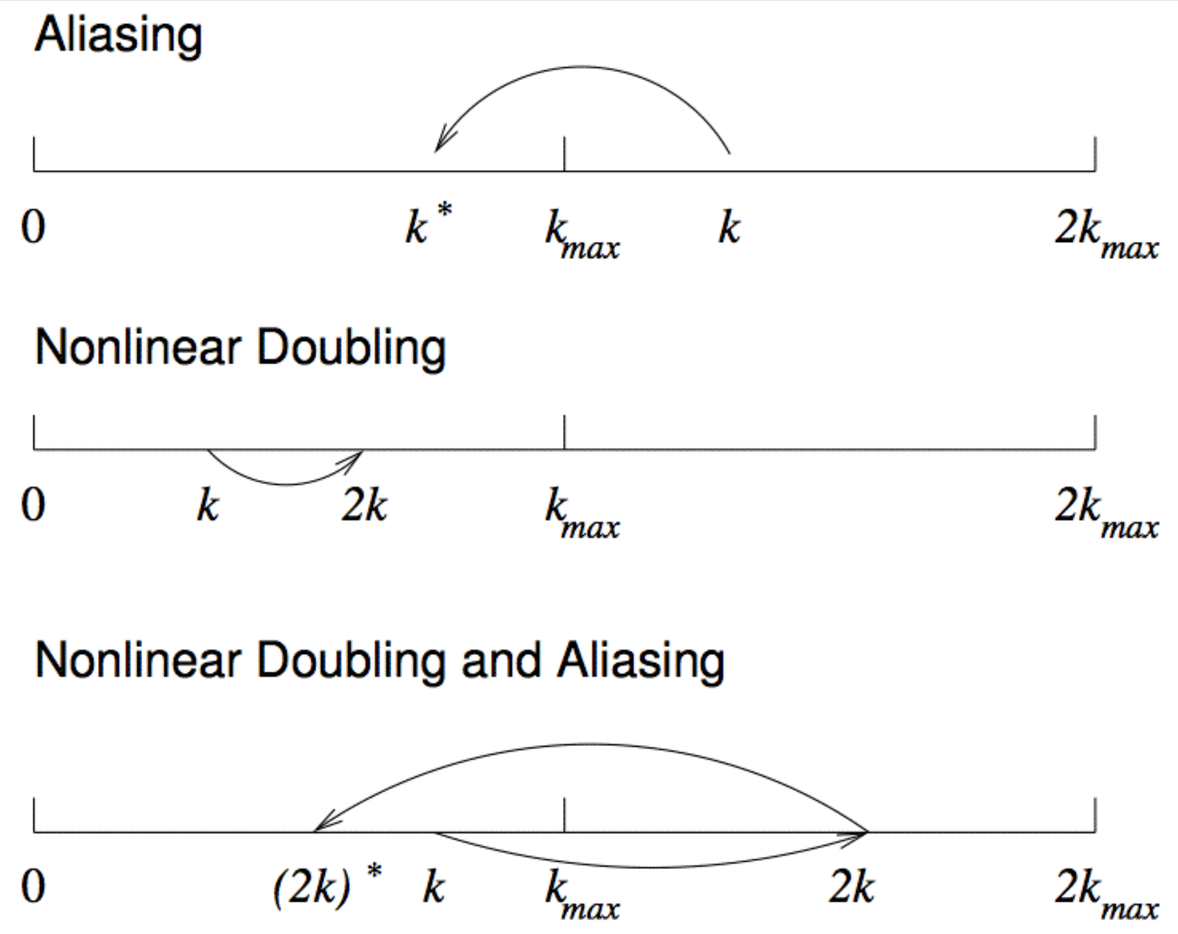
LONGER *WAVELENGTHS* ⟵

# Aliasing



www.scs.gmu.edu/climate/courses/csi756/

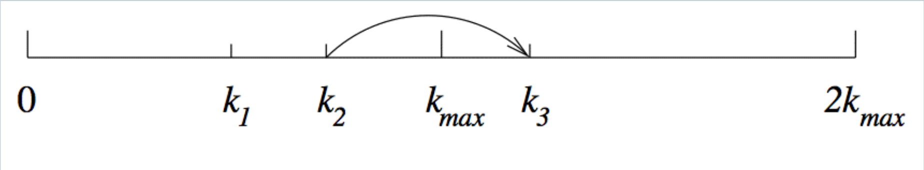
- Schopf (2005) - *Nonlinear doubling* and *aliasing*
- Energy "folded" into low *k*
- Short waves generate long wave energy



# Nonlinear instability



- If energy flows into wavenumbers *just above* $k_{max}$ ...



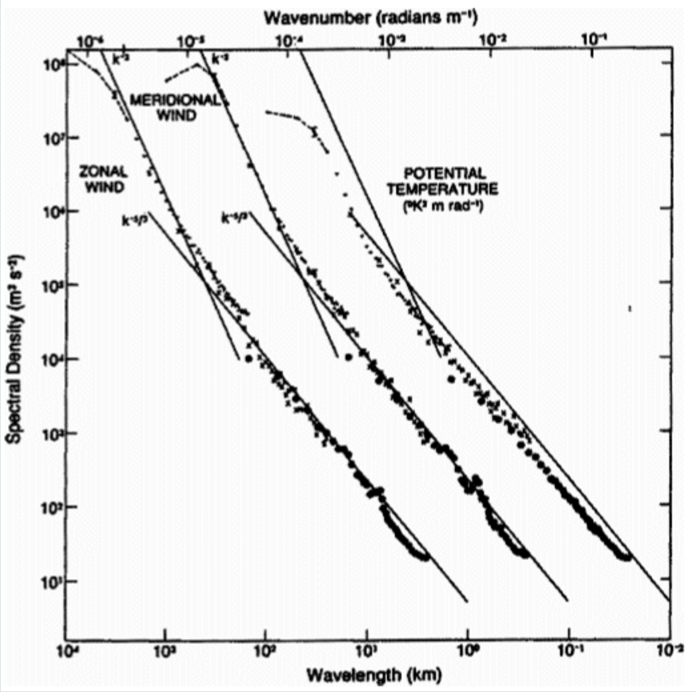

- Aliasing folds it back to wavenumbers *just below* $k_{max}$
  - builds up energy near grid resolution limit
  - Further nonlinear interaction enhances flow into wavenumbers just above $k_{max}$: accelerates process
  - This is nonlinear instability.
- What about amplitude?
  - More energy at low *k*. Say $k_2$ just below kmax ...
  - Nonlinear ($k_1$+$k_2$) has more energy *if $k_1$ small*.

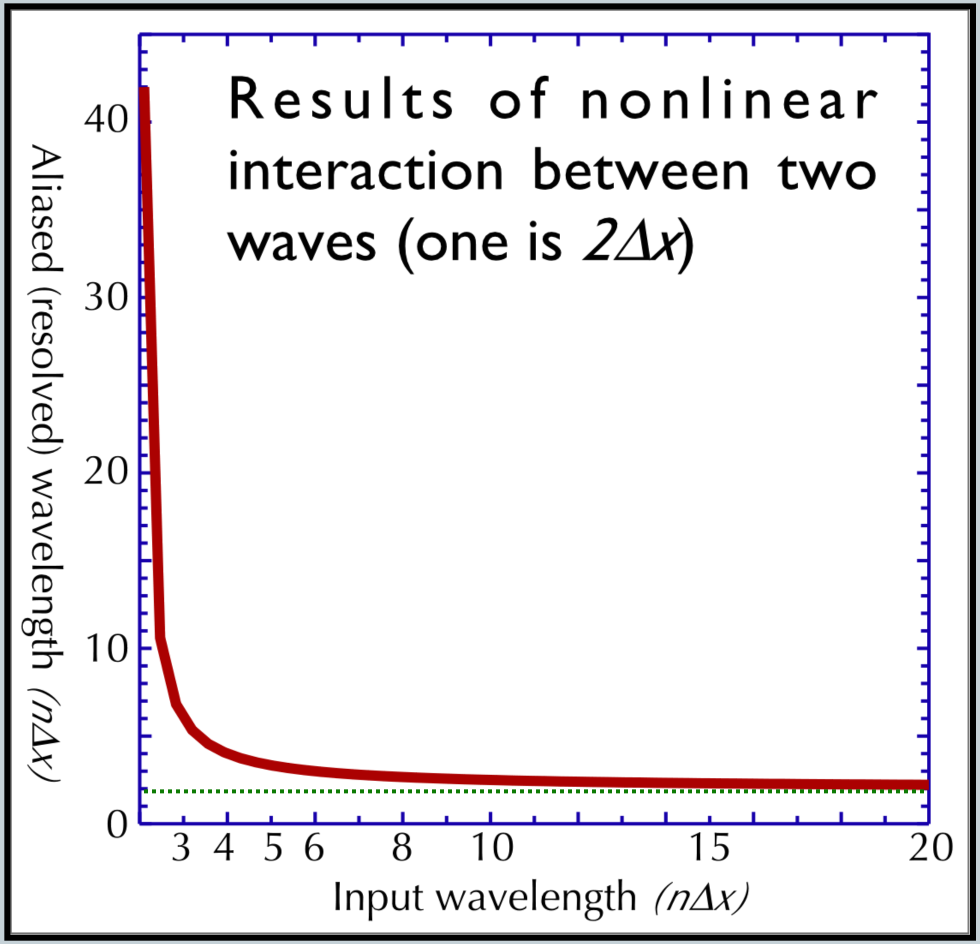
www.scs.gmu.edu/climate/courses/csi756/

# Nonlinear instability



- $2\Delta x \bullet 2.5\Delta x \longrightarrow 10\Delta x$
- $2\Delta x \bullet 8\Delta x \longrightarrow 2.7\Delta x$
- *more energy: small k*



# Aliasing in wavenumber space



| Input *wave* | Input *k / kmax* | + 2Δx = *k / kmax* | aliased *k / kmax* | = final *N*Δx |
|---|---|---|---|---|
| 2.1 Δx | 0.95 | 1.95 | 0.05 | 42 Δx |
| 5.0 Δx | 0.40 | 1.40 | 0.60 | 3.3 Δx |
| 10 Δx | 0.20 | 1.20 | 0.80 | 2.5 Δx |
| 20 Δx | 0.10 | 1.10 | 0.90 | 2.2 Δx |

# Aliasing in wavenumber space