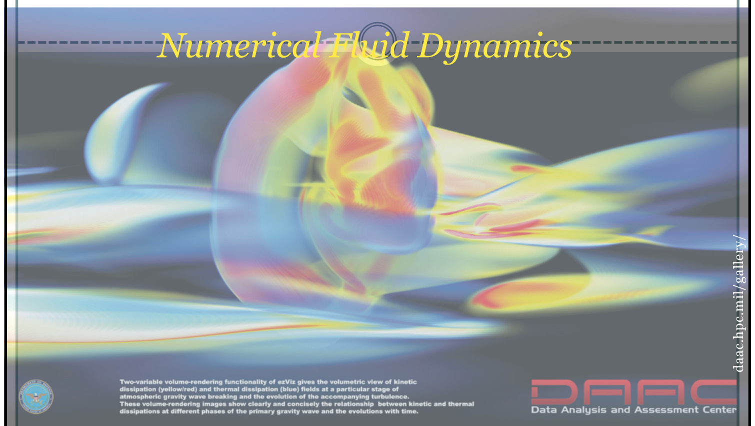
Atms 502, CSE 566

Numerical Fluid Dynamics



TUE., MAR. 26, 2019

Image: **Breaking gravity wave & turbulence**

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ATMS 502
CSE 566

Tuesday,
26 March 2019

Class #19

Plan for Today

- 1) Review
  - Visualization
  - HPC architecture
  - Amdahl's law, parallel performance
- 2) Time differencing
  - including time differencing table
- 3) Nonlinear instability & aliasing
  - problems and methods to mitigate them

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Time differencing

"STAGES" AND "LEVELS"
HIGHER ORDER ACCURACY

Terminology:
- n, n+1 … time levels
- α, β, … coefficients of spatial derivatives
- *F* approximations to spatial derivatives

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Time differencing: stages vs. levels

- *Review:* time differencing includes
  - 1) how we express the *time derivative*
  - at what *time levels* we evaluate the *spatial derivatives*
- Levels
  - … refers to how many time levels are in our scheme
  - Lax-Wendroff: *2-level*. Leapfrog: *3-level*
- Stages
  - … refers to how many times we evaluate the spatial derivatives
  - Lax-Wendroff: *single-stage*. Runge-Kutta: *2 or more stages*

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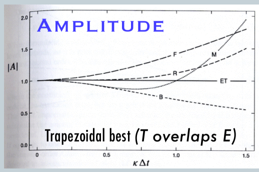
Summary: Single-stage, 2-level schemes

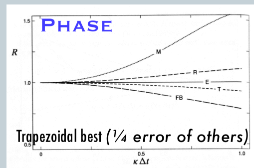
- *Single-stage*: evaluate spatial derivatives once
- 2-level: there are two time levels, *n* and *n+1*

$$\frac{\phi^{n+1}-\phi^n}{\Delta t} = \alpha F(\phi^n) + \beta F(\phi^{n+1}),\ \alpha + \beta = 1$$

Euler method: α=1,β=0
Backward method: α=0,β=1
Trapezoidal method: α=β=1/2



Trapezoidal best (*T* overlaps *E*)



Trapezoidal best (¼ error of others)

$$|A|^2 = 1 + (\alpha^2 - \beta^2)\left(\frac{\omega^2\Delta t^2}{1+\beta^2\omega^2\Delta t^2}\right)$$

$$R_{forward} = R_{backward} = 1 - \frac{(\omega\Delta t)^2}{3};\quad R_{trapezoidal} = 1 - \frac{(\omega\Delta t)^2}{12}$$

Schemes: Forward *F* • Backward *B* • Trapezoidal *T* • 2nd-order Runge-Kutta *R* • Matsuno *M*

Summary: Single-stage, 3-level schemes

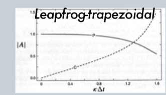
- Single-stage: evaluate spatial derivatives once
- 3-level: there are 3 time levels, *n-1, n, n+1*

$$\frac{\phi^{n+1}-(\alpha_1\phi^n+\alpha_2\phi^{n-1})}{\Delta t} = \beta_1 F(\phi^n) + \beta_2 F(\phi^{n-1})$$

$\alpha_1 = 1-\alpha_2;\ \beta_1 = \frac{\alpha_2+3}{2},\ \beta_2 = \frac{\alpha_2-1}{2}$
(β restrictions make schemes at least 2nd-order – Durran p. 58)

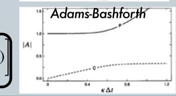
- Leapfrog: $\alpha_1=0,\ \alpha_2=1,\ \beta_1=2,\ \beta_2=0$ Time filtering; comp. mode; even/odd ..
- Leapfrog-trapezoidal A predictor-corrector method.
  2nd-order unlike time-filtered LF; computational mode damped

$$\phi^* = \phi^{n-1} + 2\Delta t F(\phi^n)$$
$$\phi^{n+1} = \phi^n + (\Delta t/2)\left[F(\phi^n) + F(\phi^*)\right]$$



- Adams-Bashforth
  2nd-order, fwd time, weak instability; computational mode damped.
  Higher-order versions of A-B exist.

$$\phi^{n+1} = \phi^n + (\Delta t)\left[\frac{3}{2}F(\phi^n) - \frac{1}{2}F(\phi^{n-1})\right]$$



Overview: Multi-stage (RK)/step methods

- Objective: higher-order accuracy in time
- Multistage: *Durran §2.3*: evaluate spatial *F* terms at several times between nΔt, (n+1)Δt
- Multistep: *Durran §2.4*: information from prior levels incorporated in integration formula.
  - Multistep: extra storage needed, but fewer evaluations of *F*
  - Multistage *or* multistep: computational modes arise.
  - Multiple forms of RK (Runge-Kutta) methods exist, e.g. low-storage form.
- More steps/or/stages: more (computational) work.
  - Good: less restrictive time step. Bad: *more* computational modes.
  - For more information: see Durran § 2.3.2, RK3 and RK4 methods.

Overview: Higher time accuracy in 3-D

- Problem: order of directional operators matters.
- Theory: *Durran §4.3*: compares exact (that from Taylor series) vs. finite difference results
  - Let's call X-operator F_1 ... and ... Y-operator F_2 ...
  - Then if $F_1F_2 = F_2F_1$, we say the *operators commute*
- Plan: if operators *don't* commute –
  - We can still get higher temporal accuracy ...
  - Use *Strang Splitting*. In 3D: *Durran eq. 4.59 p. 172* –

$$\left[F_1(\Delta t/2)\right]\left[F_2(\Delta t/2)\right]\left[F_3(\Delta t)\right]\left[F_2(\Delta t/2)\right]\left[F_1(\Delta t/2)\right]$$

Time differencing summary: Durran § 2.6

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- Durran table 2.1-2.2: summary of methods
  - *Order of scheme*
  - *Storage factor*
    - *Number of full arrays needed*
    - *Not given if implicit; depends on method*
  - *Efficiency factor*
    - *Largest stable step ÷ by # evaluations of **F***
  - *Max s*
    - *largest stable $\omega\Delta t$*

$h=\Delta t$; $s=\omega\Delta t$!!

Table 2.1 Summary of methods for the solution of ordinary differential equations. The second- and third-order Runge–Kutta methods are low-storage variants; $h = \Delta t$

Method	Order	Formulae
Forward	1	$\phi_{n+1} = \phi_n + hF(\phi_n)$
Backward	1	$\phi_{n+1} = \phi_n + hF(\phi_{n+1})$

DURRAN 2ND ED. TABLES 2.1-2.2, PP 83-84

Method	Storage factor	Efficiency factor	Amplification factor	Phase error	Max s
Forward	2	0	$1+\frac{s^2}{2}$	$1-\frac{s^2}{3}$	0

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Aliasing, nonlinear instability, and conservation

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REFERENCES:
DURRAN SECTION § 4.4.1, 4.5
ROBERT WILHELMSON NOTES
HALTINER AND WILLIAMS SECTION 5-11-1
PAUL SCHOPF NOTES, SCHOOL OF COMPUTATIONAL SCIENCES, GEORGE MASON UNIVERSITY (MASON.GMU.EDU)

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(Nonlinear) advection

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- Back to the familiar.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \text{- or -} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

  - The problem for finite differences lies in the multiplication.*

- Let u=sin(kx); then

$$u = \sin kx \;\Rightarrow\; \frac{\partial}{\partial x}\left(u^2\right) = k\sin 2kx$$

  - What effective wavenumber are we working with now?

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Aliasing

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- Even if u=sin*kx* is resolved…
  - If … (kmax/2) < k < kmax,
  - the nonlinear term puts energy into:

    kmax < k < 2kmax

  - This is not resolvable!
    - ✓ It will appear in other, resolvable wavelengths.
    - ✓ This is aliasing.

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Aliasing: Example

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- Some studies have investigated gravity waves near weather fronts.
- They appeared in observations
- They appeared in high-resolution simulations.
- Some of the *modeled* waves were not real.
- They appeared at *high horizontal resolution.*

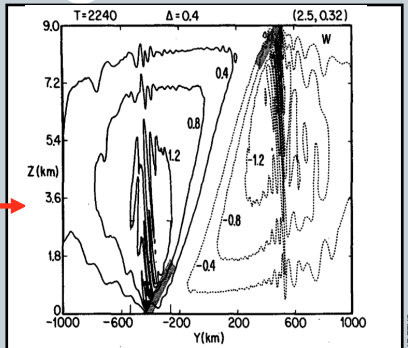

FIG. 2. Vertical velocity at $T = 2240$ min from the beginning of the experiment. The experiment that produced this figure had a horizontal resolution of 2.5 km and a vertical resolution of 320 m. This

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Spurious waves

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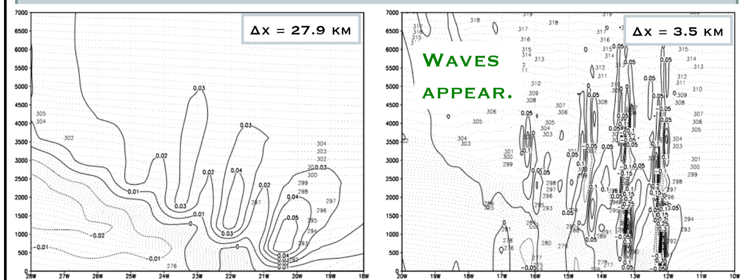

- Iga (2005) - waves on a front. Vertical velocity is shown.
- Only change is decrease in Δx.

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Spurious waves

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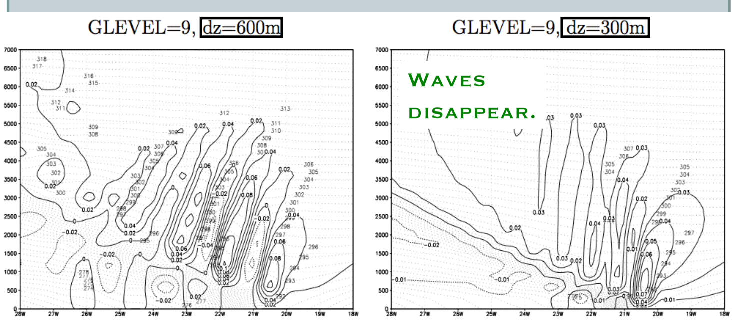

- Iga (2005) - waves on a front. Vertical velocity is shown. Δx=14 km. Only change is decrease in Δz.

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Spurious waves

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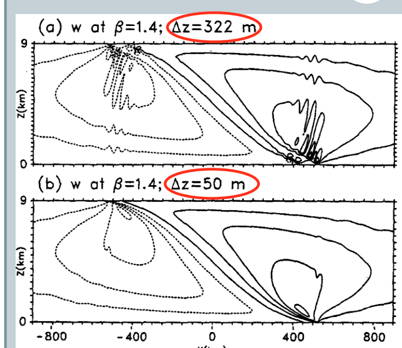

- Snyder et al. 1993. Only change is Δz.
- Waves appear at high *horizontal* resolution
- Waves disappear at increased *vertical* resolution.
- *Why?*

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Spurious waves

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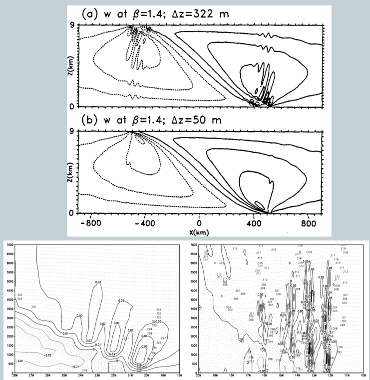

One possibility…

- Reducing Δx while keeping Δz constant
- .. results in small horizontal waves with small vertical scales … too small to be resolved.
- Uh-oh!

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Aliasing

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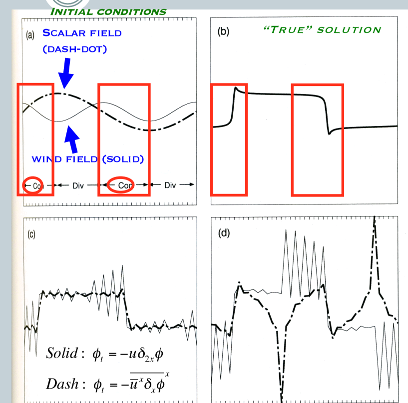
- Simple differential (t) - difference (x) equation results
- Problem: linear equation, variable coefficients
  - in other words, c=c(x), but not (t)
- Small scales grow preferentially



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Aliasing

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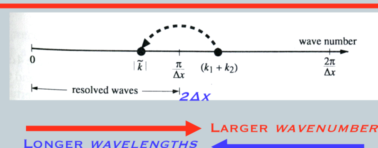
- Where do unresolved waves "go" ?
  - *Durran eqn. 3.91:*

$$\tilde{k} = \begin{cases} k_1 + k_2 - \dfrac{2\pi}{\Delta x}, \text{ if } k_1 + k_2 > \dfrac{\pi}{\Delta x} \\ k_1 + k_2 + \dfrac{2\pi}{\Delta x}, \text{ if } k_1 + k_2 < \dfrac{-\pi}{\Delta x} \end{cases}$$

  - *Examples:*
    - 2Δx • 2.5Δx → 10Δx
    - (4/3)Δx → 4Δx
  - Note if *both* waves are 4Δx or longer: *no aliasing*



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Aliasing

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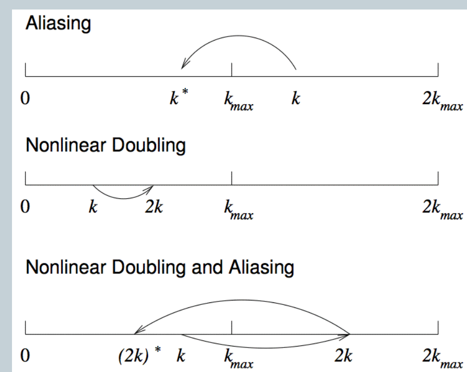
www.scs.gmu.edu/climate/courses/csi756/

- Schopf (2005) - *Nonlinear doubling* and *aliasing*
- Energy "folded" into low k
- Short waves generate long wave energy



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Nonlinear instability

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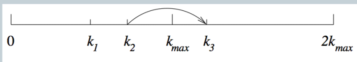
- If energy flows into wavenumbers *just above* k_{max} …



- Aliasing folds it back to wavenumbers *just below* k_{max}
  - builds up energy near grid resolution limit
  - Further nonlinear interaction enhances flow into wavenumbers just above k_{max}: accelerates process
  - This is nonlinear instability.
- What about amplitude?
  - More energy at low *k*. Say k_2 just below kmax …
  - Nonlinear (k_1+k_2) has more energy *if* k_1 *small*.

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Nonlinear instability

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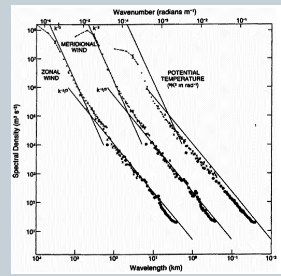
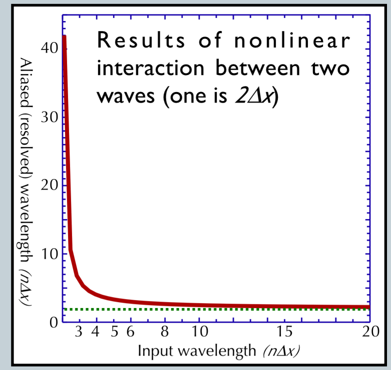
- 2Δx • 2.5Δx → 10Δx
- 2Δx • 8Δx → 2.7Δx
- *more energy: small k*





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Aliasing in wavenumber space

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Input wave	Input k / kmax	+ 2Δx = k / kmax	aliased k / kmax	= final NΔx
2.1 Δx	0.95	1.95	0.05	42 Δx
5.0 Δx	0.40	1.40	0.60	3.3 Δx
10 Δx	0.20	1.20	0.80	2.5 Δx
20 Δx	0.10	1.10	0.90	2.2 Δx

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Aliasing in wavenumber space

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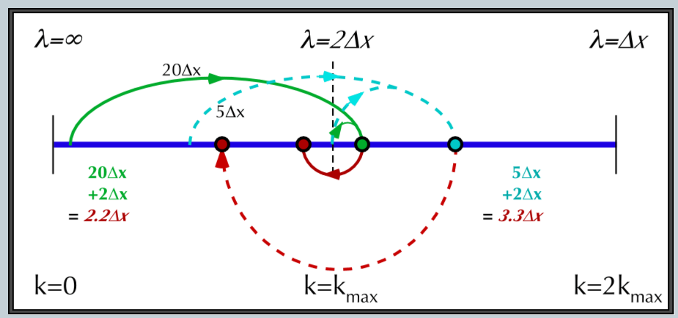

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