Atms 502
CSE 566

Numerical Fluid Dynamics

Tue., Feb. 19, 2019

ATMS 502
CSE 566

Tuesday,
19 February 2019

Class #11

Class 10 was for exam #1

Plan for Today

- 1) REVIEW
  - Smolarkiewicz, diagonal flow, splitting
- 2) NUMERICAL METHODS :
  - The origin of Lax-Wendroff
  - Fully implicit schemes
- 3) NUMERICAL METHODS :
  - Introduction to nesting

Numerical methods: Lax-Wendroff

References:
- C001 (Lax-Wendroff)
- C006 (Finite differences)
- C007 (Taylor series)
- C052 (Advection)

Where is 1-D Lax-Wendroff *from*?

- Lax-Wendroff a.k.a. 2nd-order Crowley
  - Taylor series expansion for ϕ^{n+1}
    - where $\phi(t+\Delta t)$ leads … with extra term on RHS

$$\phi^{n+1} = \phi^n + \Delta t \phi_t + \frac{\Delta t^2}{2}\phi_{tt} + \ldots$$

$$\textit{Substitute}: \ \phi_t = -c\phi_x, \ \phi_{tt} = c^2\phi_{xx}$$

$$\phi^{n+1} = \phi^n + \Delta t\left(-c\phi_x\right) + \frac{\Delta t^2}{2}\left(c^2\phi_{xx}\right)$$

$$= \phi^n - c\Delta t\phi_x + \frac{c^2\Delta t^2}{2}\phi_{xx}$$

Using centered differences works here - this is what we are using in program #1.

Review: Lax-Wendroff

- Lax-Wendroff: Taylor series, 1-way wave eqn.

must approximate these derivatives

$$\phi(t+\Delta t)=\phi(t)+\Delta t\frac{\partial\phi}{\partial t}+\frac{(\Delta t)^2}{2!}\frac{\partial^2\phi}{\partial t^2}+\ldots \text{ (change notation...)}$$

$$=\phi(t)+\Delta t\phi_t+\frac{(\Delta t)^2}{2!}\phi_{tt}+\ldots \text{ ; neglect terms past } \phi_{tt} \text{ for Lax-W.}$$

1-way wave equation: $\phi_t=-c\phi_x$ takes care of first time derivative.

For ϕ_{tt}, $\phi_t=-c\phi_x \Rightarrow \phi_{tt}=-c\phi_{xt}$ and:

$$\phi_t=-c\phi_x \Rightarrow \phi_{xt}=-c\phi_{xx} \text{ so } \phi_{tt}=-c(-c\phi_{xx})=c^2\phi_{xx}; \textit{ Thus,}$$

$$\phi(t+\Delta t)=\phi(t)+\Delta t(-c\phi_x)+\frac{(\Delta t)^2}{2!}(c^2\phi_{xx}); \text{ use centered derivatives to finish:}$$

$$\phi(t+\Delta t)=\phi(t)-c\Delta t\left(\frac{\phi_{j+1}-\phi_{j-1}}{2\Delta x}\right)+c^2\frac{(\Delta t)^2}{2!}\left(\frac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{(\Delta x)^2}\right)$$

Notes

Fully Implicit Schemes

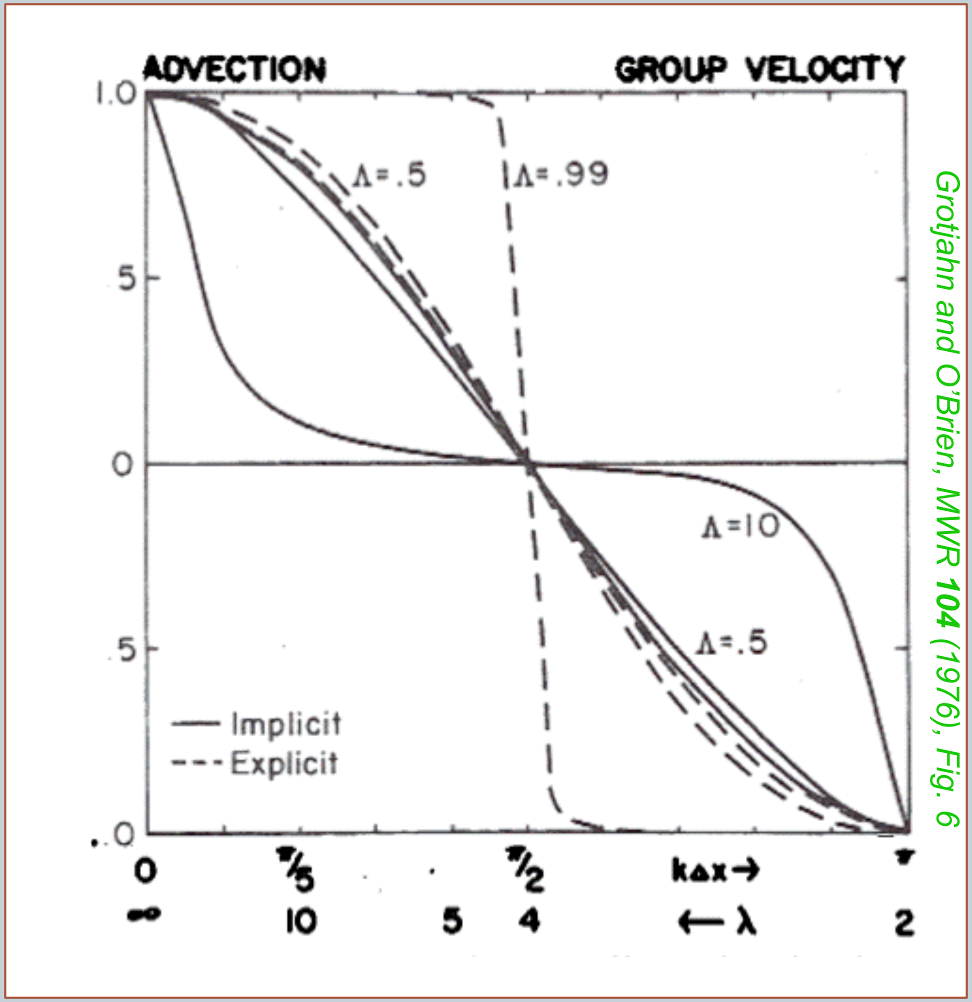
Reference pages for this section:

- C002 – explicit numerical methods
- C070 – implicit numerical methods

Explicit vs. Implicit: advection

At right:

- <u>Group velocities</u> for 1-D advection for explicit (dashed) and implicit (solid)
- <u>Explicit</u> always more accurate for waves longer than 4Δx



Grotjahn and O'Brien, MWR **104** (1976), Fig. 6

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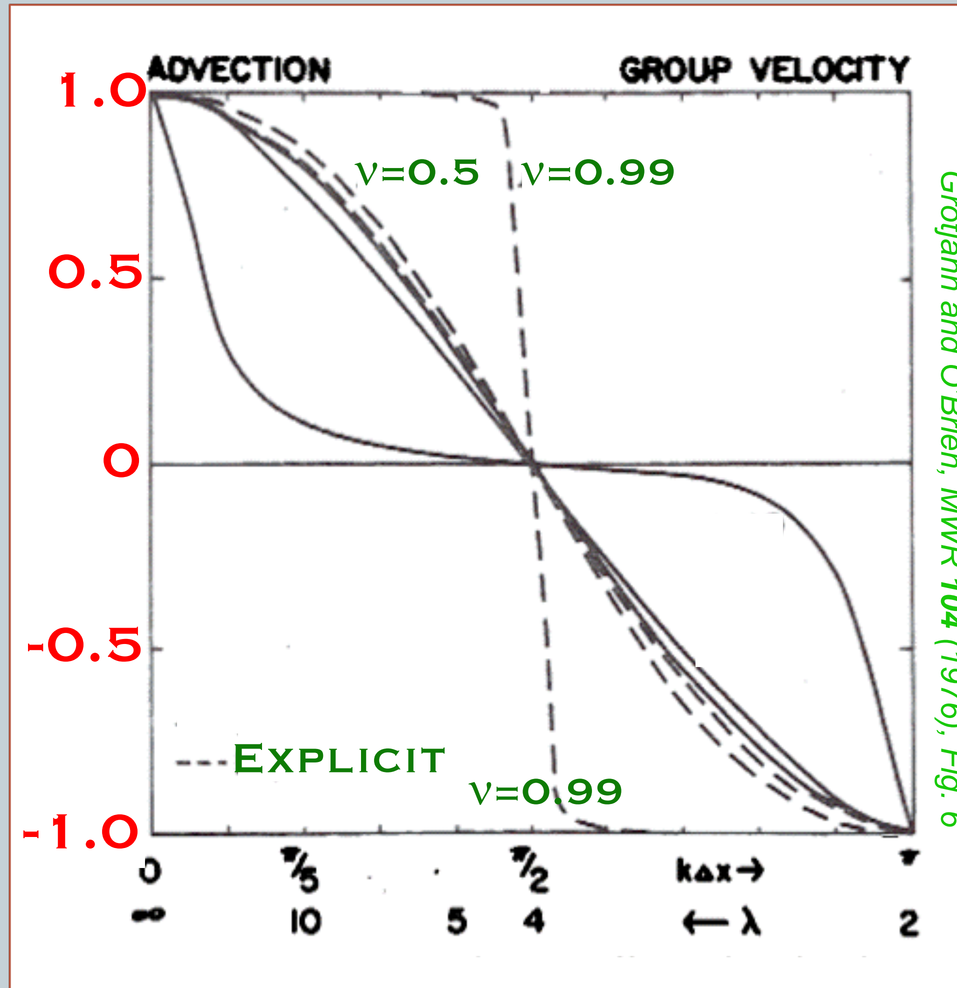

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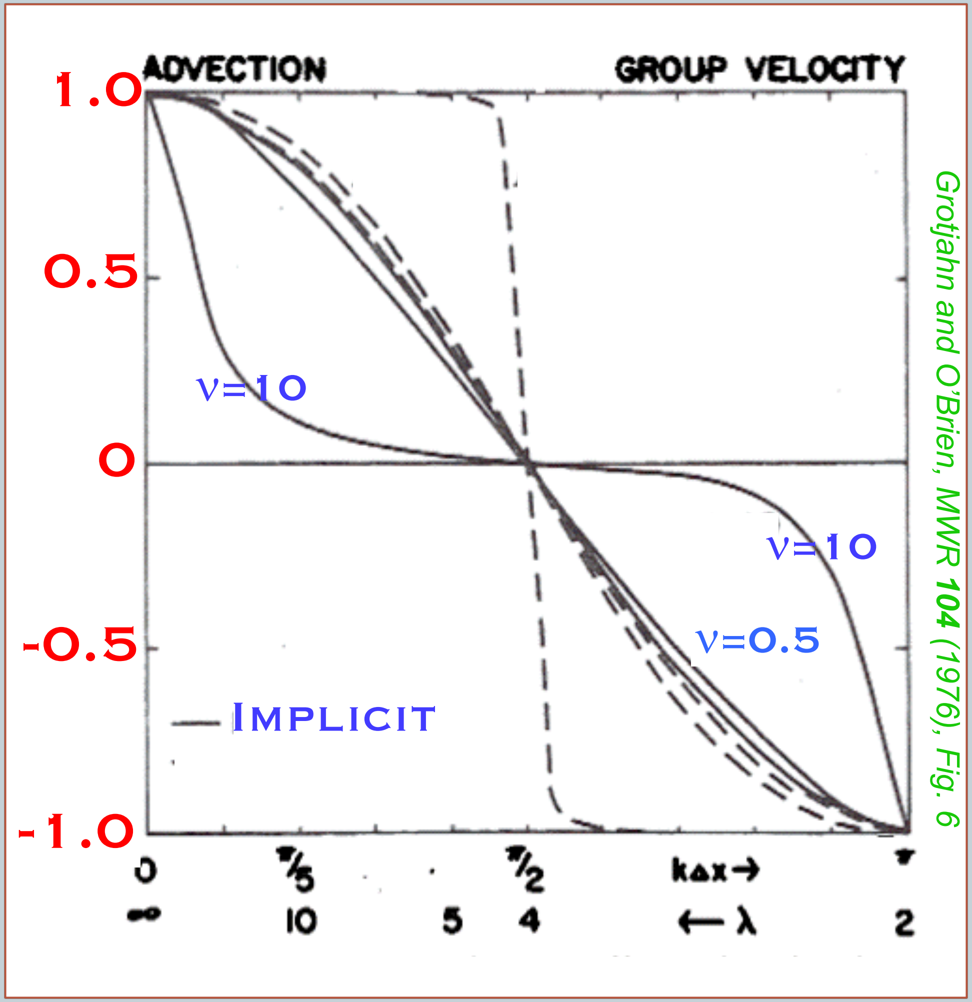

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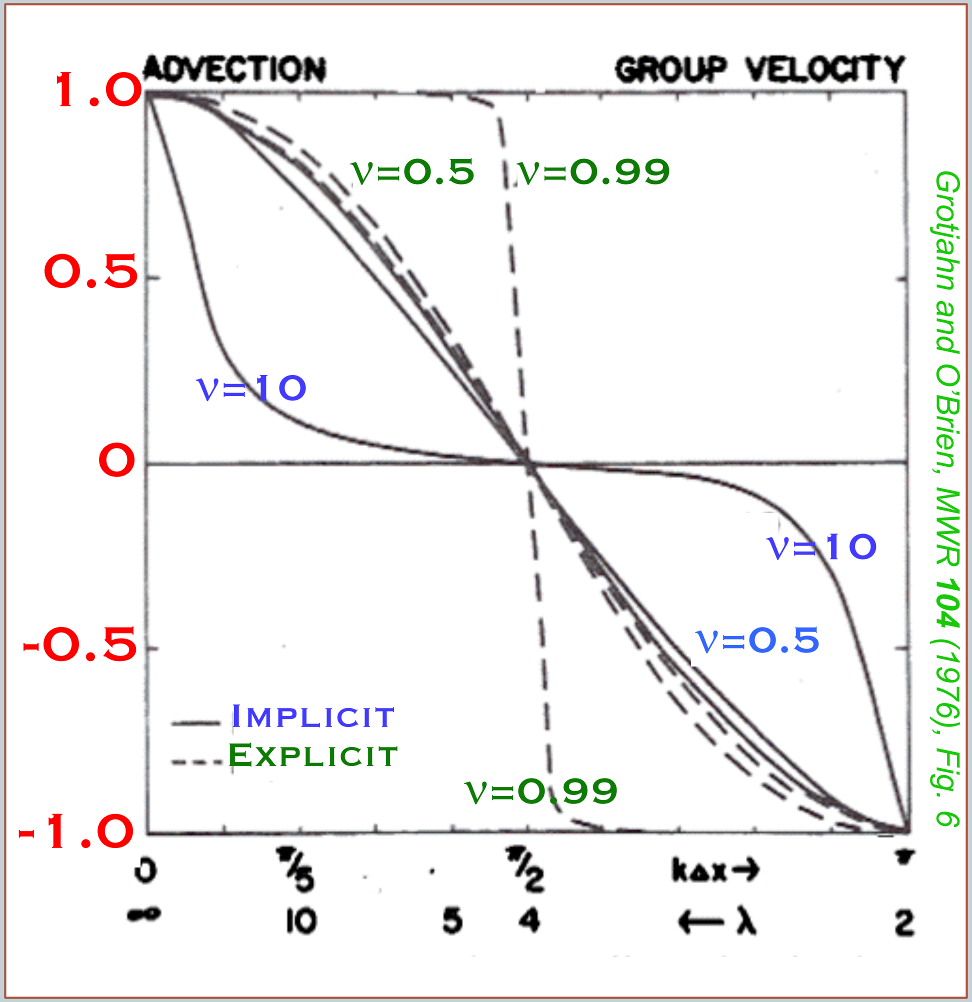
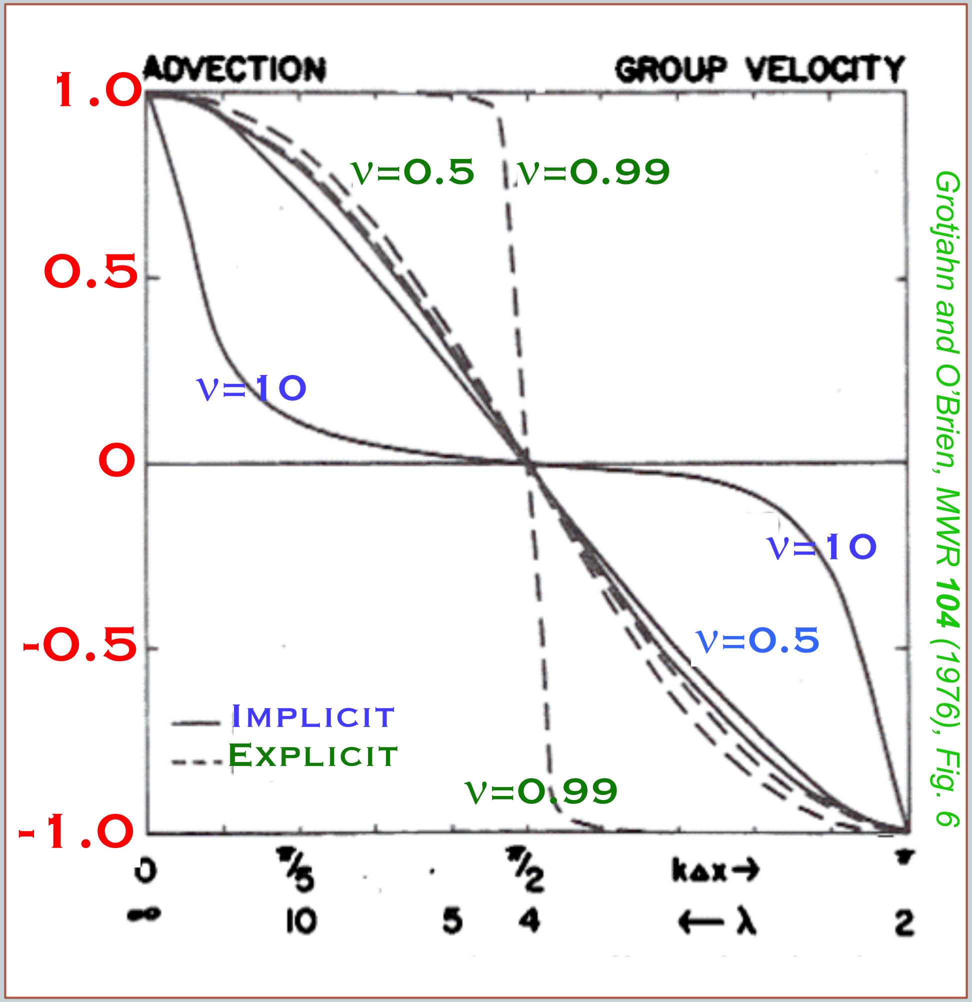


Explicit vs. Implicit: advection

At right:

- Group velocities for 1-D advection for explicit *(dashed)* and implicit *(solid)*
- Explicit more accurate for waves longer than 4Δx
- Implicit allows large time step but has large errors, as seen here for ν=10
- Group velocities always negative for 2Δx



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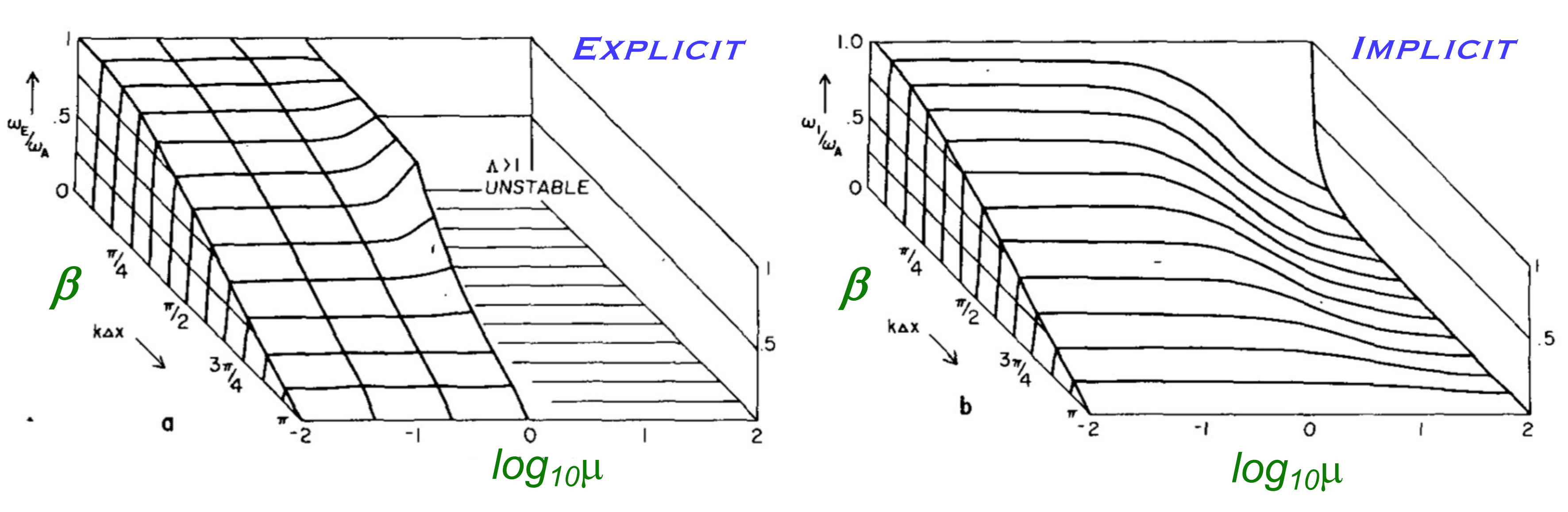
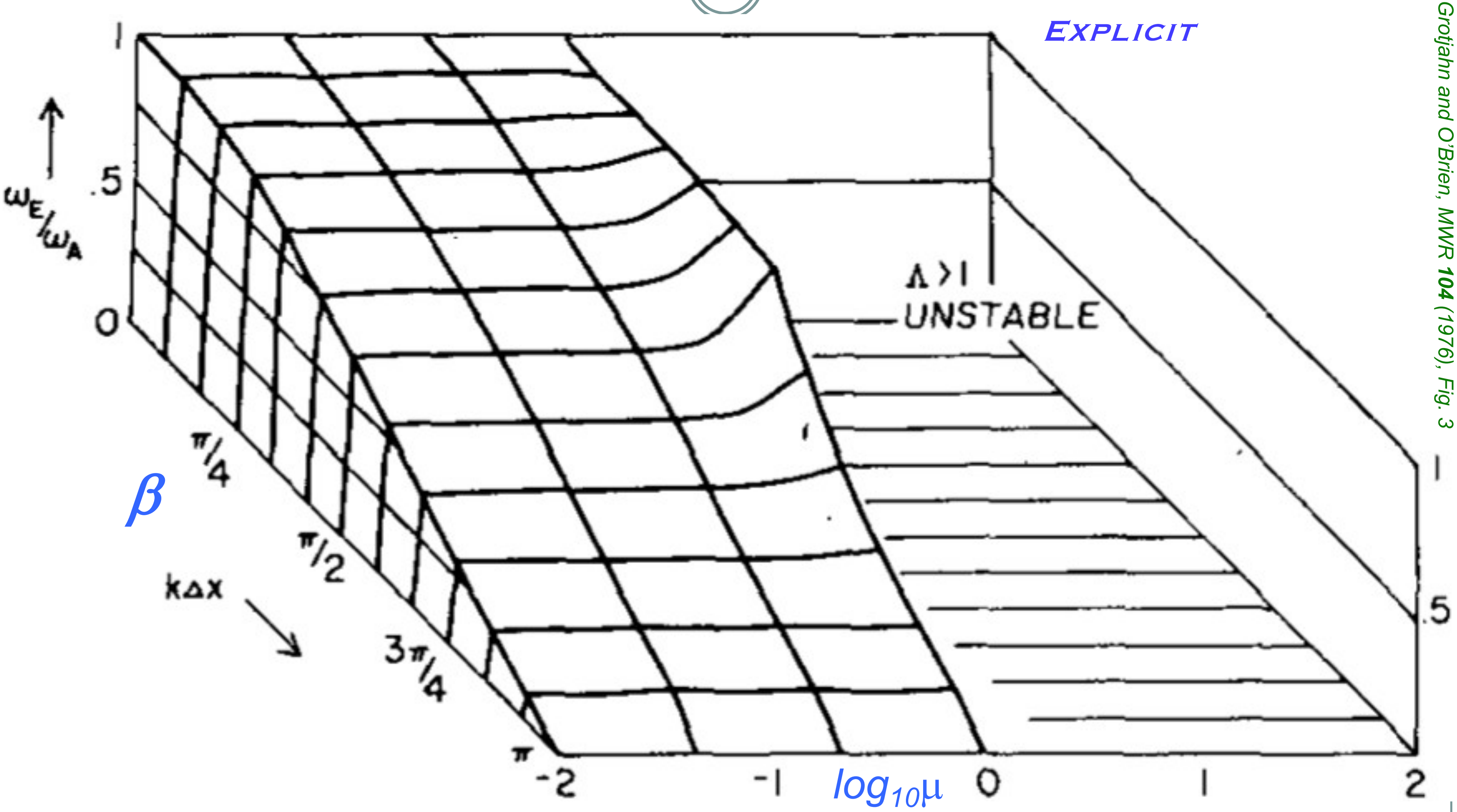


Grotjahn and O'Brien, MWR **104** (1976), Fig. 3

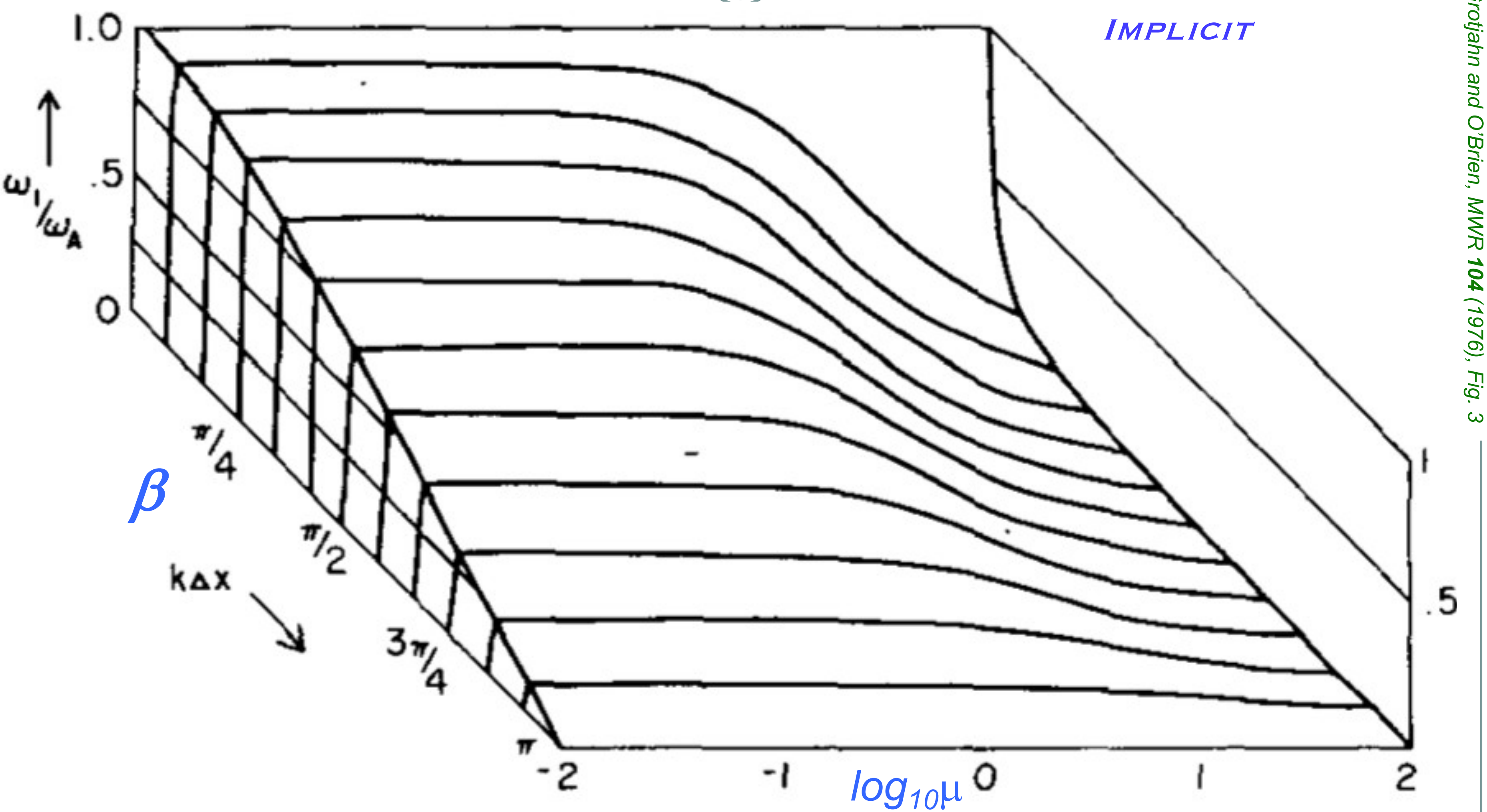
- Relative phase velocity shown in 3D for implicit and explicit schemes
- $\mathrm{Log}_{10}(\mu)$ on abscissa; $\beta = k\Delta x$ on ordinate.
- Implicit schemes maintain stability by *slowing down the waves*.

Explicit advection



Grotjahn and O'Brien, MWR **104** (1976), Fig. 3

Implicit advection



Grotjahn and O'Brien, MWR **104** (1976), Fig. 3

Implicit Schemes: FTCS

- Recall the forward time, centered space (FTCS) scheme applied to the linear advection equation..

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta t}$$

  - ... has amplification factor: $\lambda = 1 - i\mu \sin\beta$
  - ... and is thus unstable.

- Stabilize this scheme using du/dx at *(n+1)*:

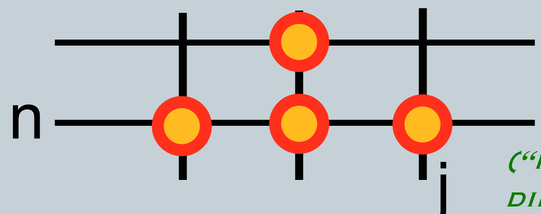
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$

Implicit Schemes: FTCS

- Explicit

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}$$

  - Computational molecule



(“FORWARD” DIFFERENCING)

  - Amplification factor

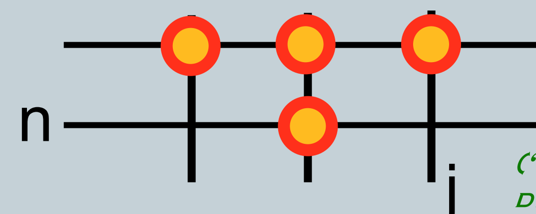
$$\lambda = 1 - i\mu\sin\beta$$

(UNSTABLE)

- Fully implicit

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$

  - Computational molecule



(“BACKWARD” DIFFERENCING)

  - Amplification factor

$$\lambda = \frac{1 - i\mu\sin\beta}{1 + \mu^2\sin^2\beta}$$

(STABLE)

Both are $O(\Delta t, \Delta x^2)$ accurate. Explicit is unstable; Implicit is unconditionally stable but has large errors for large time steps, and requires matrix inversion.

Implicit Schemes: FTCS

- Euler scheme:

$$u_j^{n+1} - u_j^n = -(\mu/2)\left(u_{j+1}^{n+1} - u_{j-1}^{n+1}\right)$$

- Coefficients of each term:

$$\left(-\frac{\mu}{2}\right)u_{j-1}^{n+1} + (1)u_j^{n+1} + \left(\frac{\mu}{2}\right)u_{j+1}^{n+1} = u_j^n$$

$$au_{j-1}^{n+1} + bu_j^{n+1} + cu_{j+1}^{n+1} = D$$

- Solve set of equations
  - For all *j*, simultaneously
  - Matrix [A] is *tridiagonal*
    - rapid solvers exist

- Solution

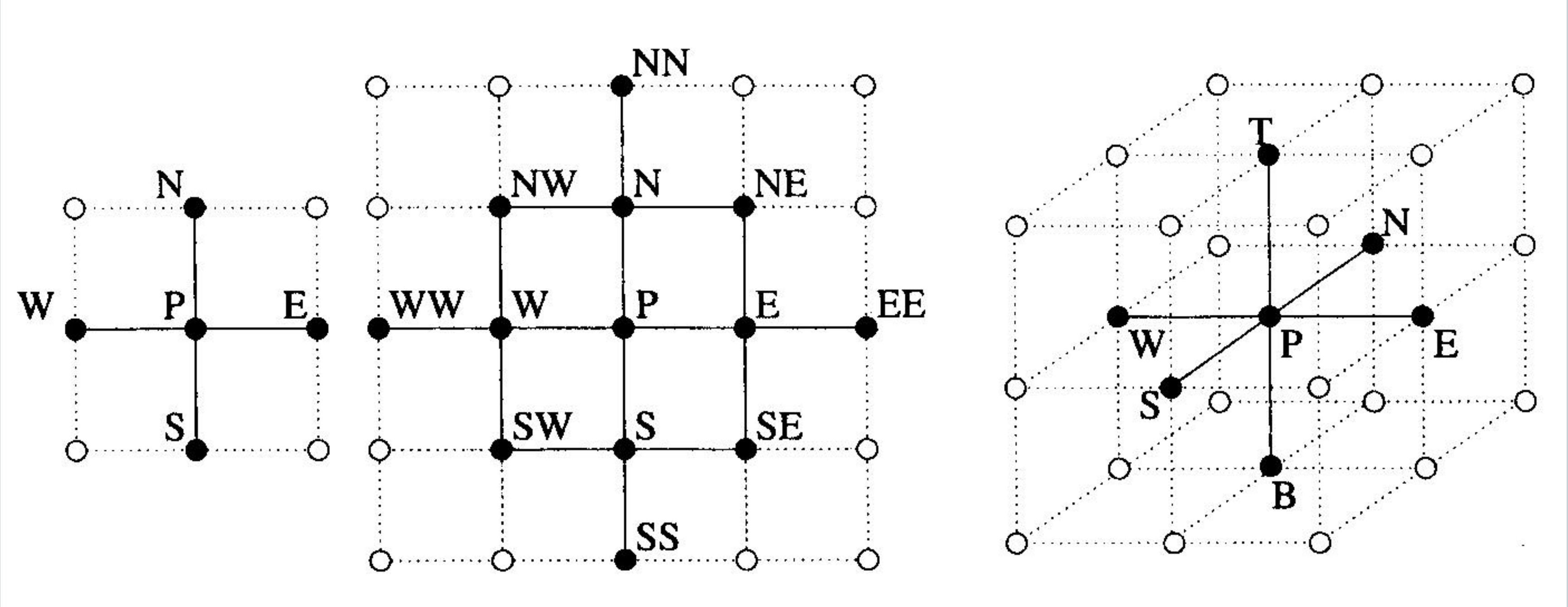
[A] [u] [D]

$$\begin{bmatrix} b_1 & c_1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ a_2 & b_2 & c_2 & & & & & & \cdot \\ 0 & a_3 & b_3 & c_3 & & & & & \cdot \\ \cdot & & \ddots & \ddots & \ddots & & & & \cdot \\ \cdot & & & \ddots & \ddots & \ddots & & & \cdot \\ \cdot & & & & \ddots & \ddots & \ddots & & 0 \\ \cdot & & & & & & a_{M-1} & b_{M-1} & c_{M-1} \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_M & b_M \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_{M-1}^{n+1} \\ u_M^{n+1} \end{bmatrix} = \begin{bmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D_{M-1} \\ D_M \end{bmatrix}$$

- *M* actual grid points: j=1,…,M
- Boundary values are specified at j=0 and j=M
- *M*+2 points (total)
- Initial conditions known at n=0

Implicit schemes

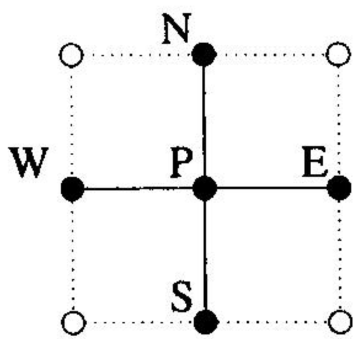
- There are differencing schemes requiring more points - either for higher order schemes, or more dimensions.



Ferziger p. 55

Implicit schemes

- The matrix layout changes appropriately.



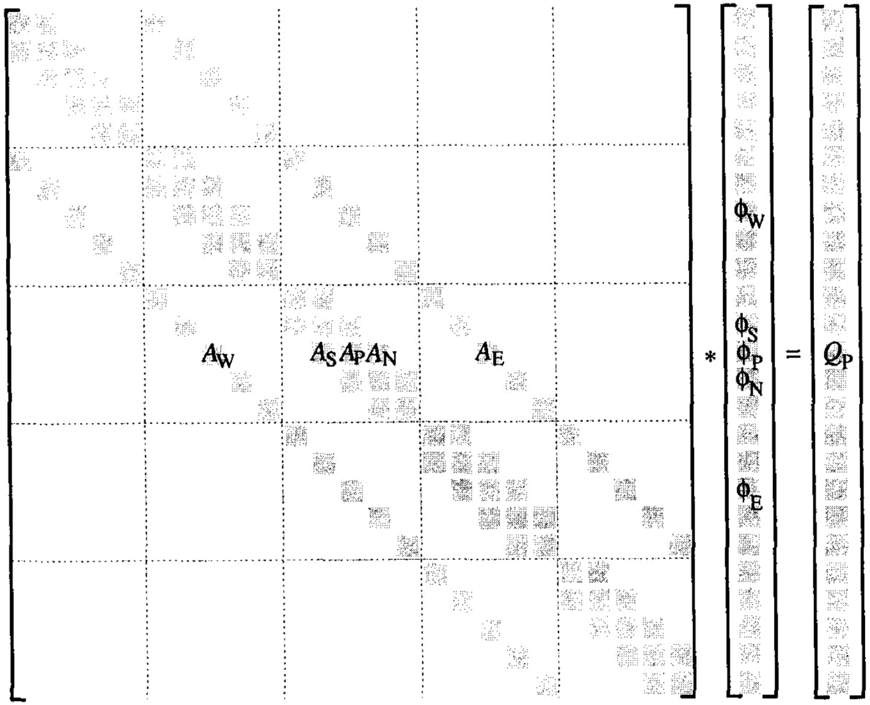


Fig. 3.5. Structure of the matrix for a five-point computational molecule (non-zero entries in the coefficient matrix on five diagonals are shaded; each horizontal set of boxes corresponds to one grid line)

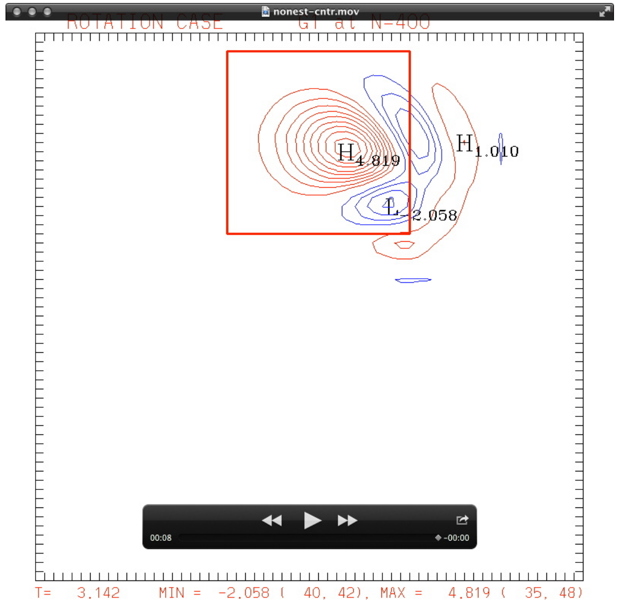
Ferziger pp. 55, 57

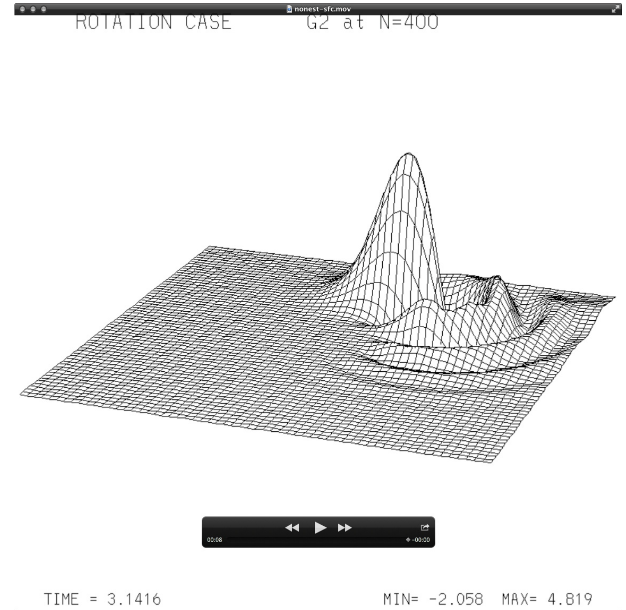
Nesting: introduction

THE BIG PICTURE.

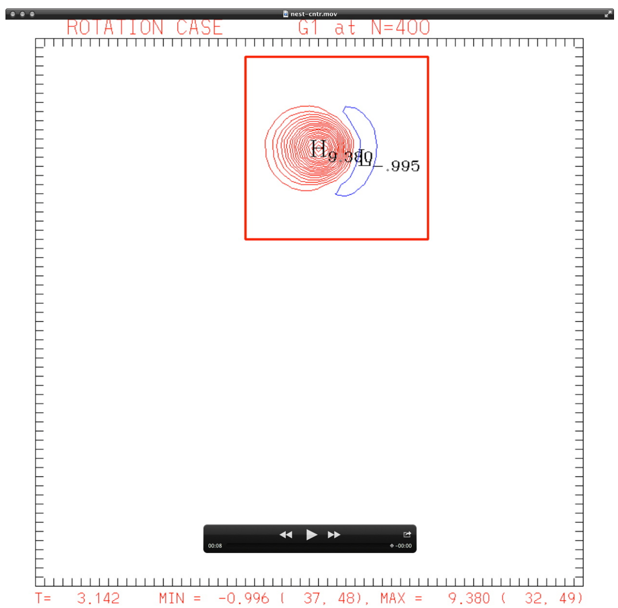
After 1 rotation

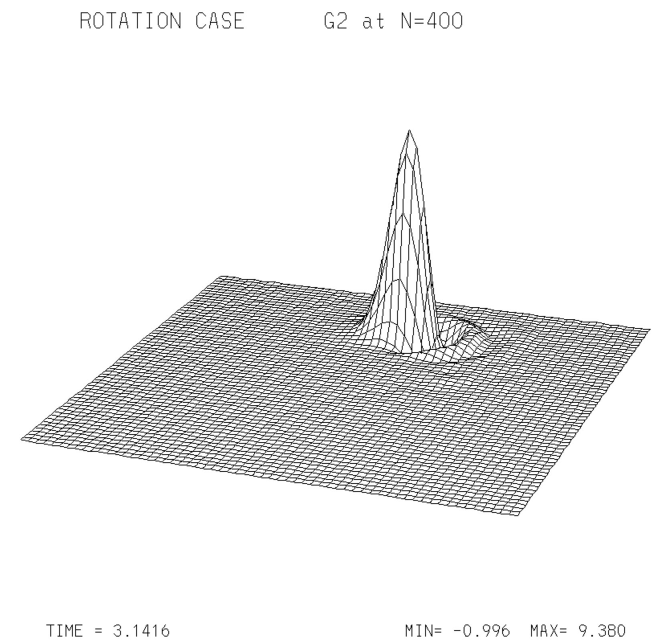
No nest





Nest



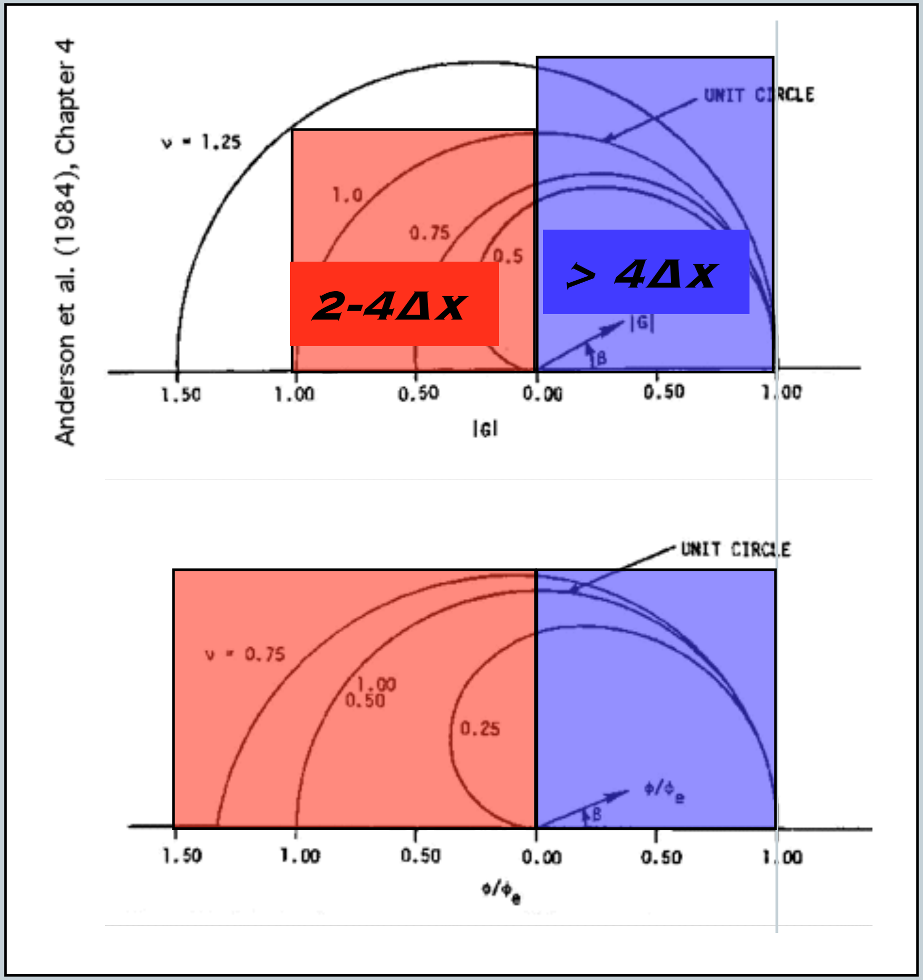


Why are we nesting

- To minimize errors -
  - Amplitude
  - Phase speed
  - Group velocity
  - Truncation error
- To better resolve small-scale features
  - … which are damped out if represented by too few grid points (e.g. less than 4-6Δx)
- To save computation time
  - Nested grids allow higher resolution only where needed

Why are we nesting

- To minimize errors -
  - Amplitude
  - Phase speed
  - Group velocity
  - Truncation error
- To better resolve small-scale features
- To save computation time
- Nesting: "move" solution to lower wavenumbers.



Irregular grids

- Instead of regular, structured grids -

"Method of lines" - Grid points reposition themselves during integration

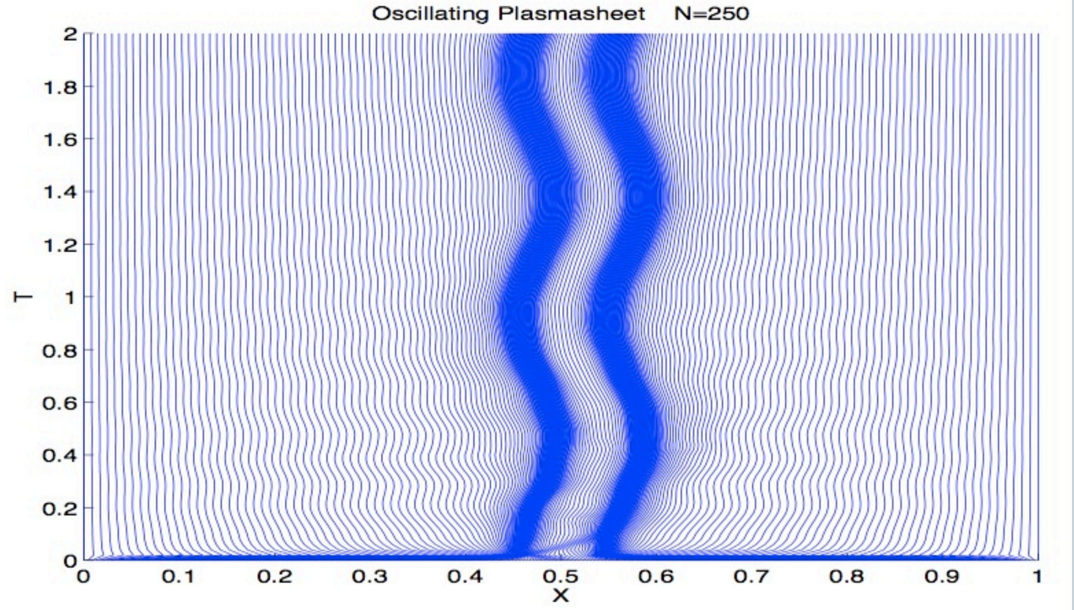


Fig. 1. The grid history in a 1D MHD simulation of an oscillating plasma sheet embedded in a vacuum. Starting with an equidistant grid of 250 grid points, the sheet boundaries are automatically recognized as regions where grid points need to be clustered. After this rapid initial adjustment (prior to times $T < 0.05$), the mesh clearly follows the oscillation.

"Dynamic grid adaptation for computational magnetohydrodynamics" - *Keppens et al., 2000*

Irregular grids

- Instead of regular, structured grids -

Grid distortion in response to evolving solution

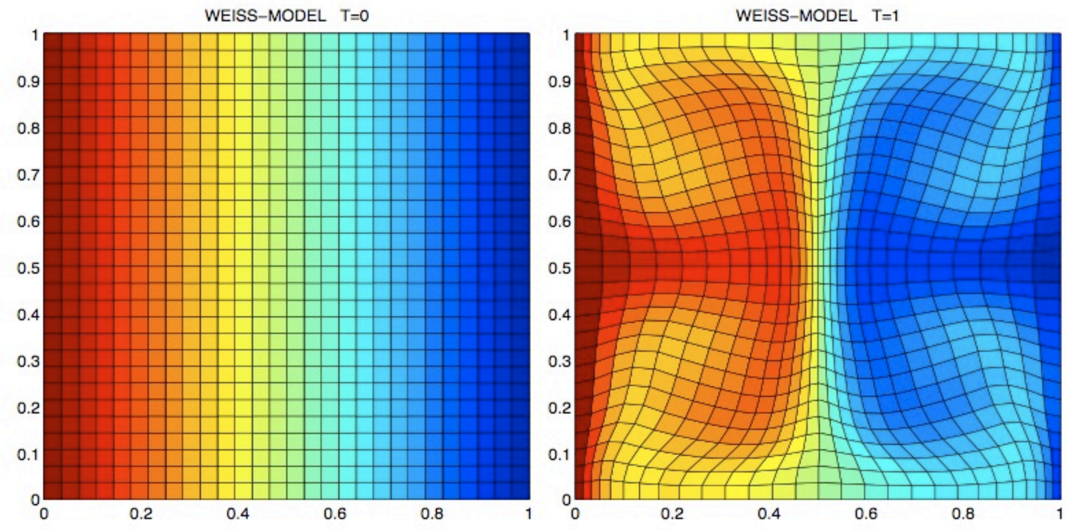


Fig. 2. A 2D kinematic flux expulsion. The left panel shows the initial cartesian mesh and the shading corresponds to the magnetic vector potential. Right panel: an imposed four-cell convection pattern causes the initially straight, uniform field to distort, which is recognized and followed by the 2D grid cell movements.

"Dynamic grid adaptation for computational magnetohydrodynamics" - *Keppens et al., 2000*

2D nesting

Evolving 2D grid

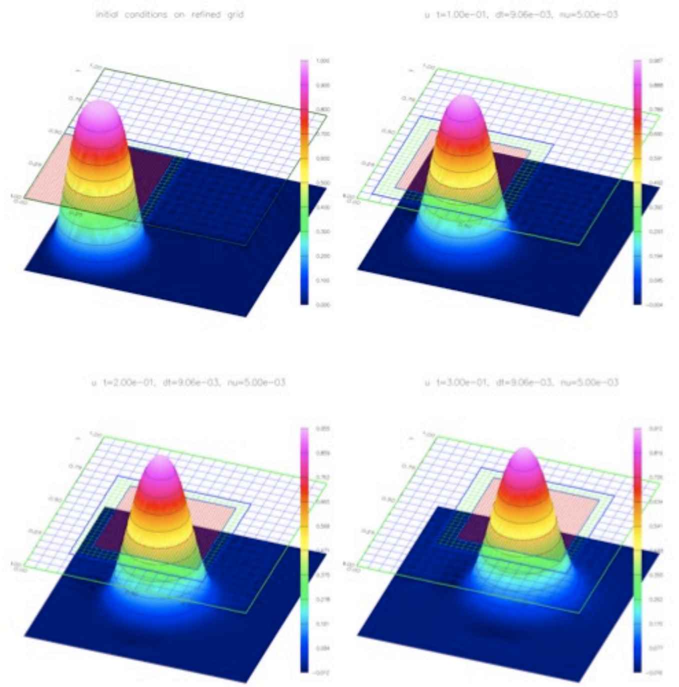
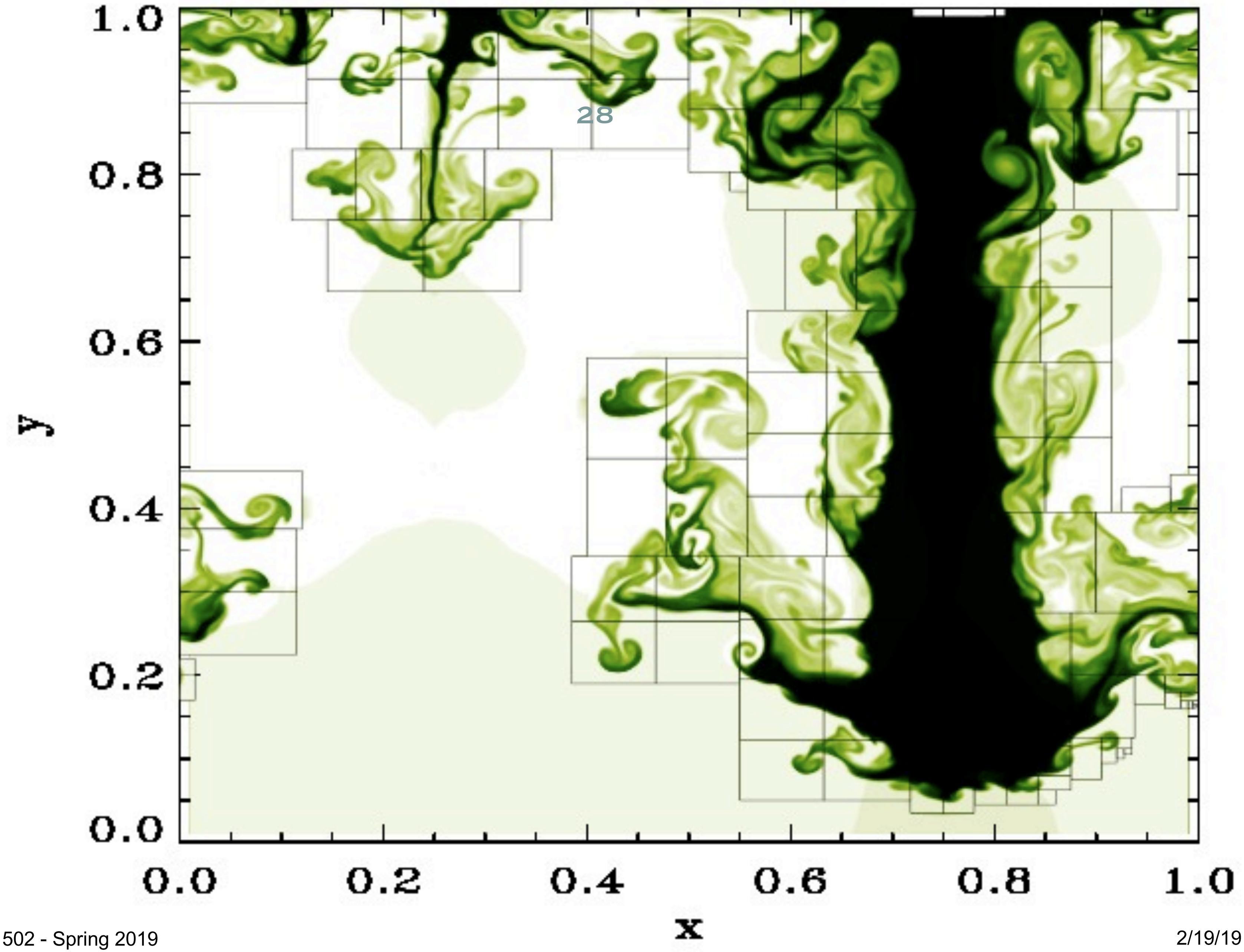


Figure 1: Results from hypeAmr, solving a convection diffusion equation with adaptive mesh refinement.

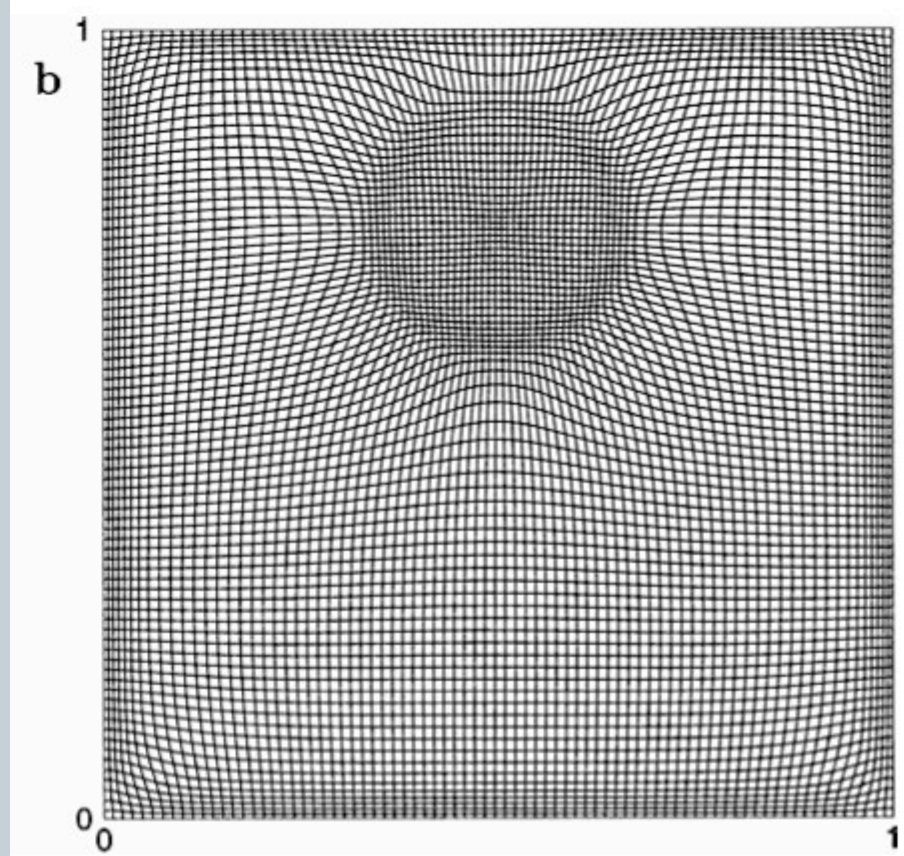
“Adaptive mesh refinement routines for Overture” -
Brown and Henshaw, 2003

"Adaptive Mesh Refinement" - *Keppens*



Irregular grids

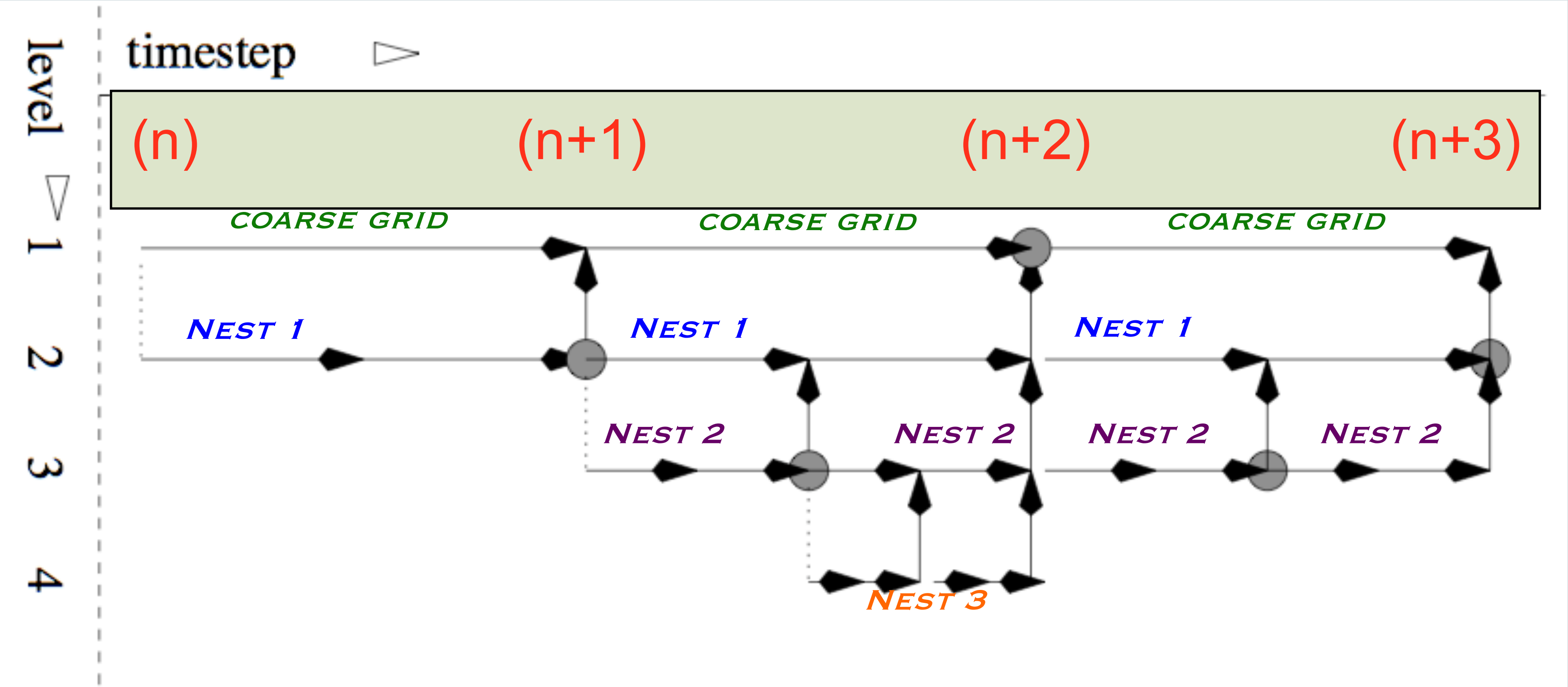
2D gridpoint redistribution



0 × 80 grid points with $\lambda_v = 4.0$, $\lambda_o = 0.0$, and $K = 4$ smoothing
(b) Improved quality grid.

“Dynamic grid adaptation using the MPDATA scheme” -
Iselin et al., 2002

(Time) stepping forward when nesting



KEPPENS ET AL., 2000

Nesting: Implementation

COMPUTER PROBLEM #4

Program #4: main routine