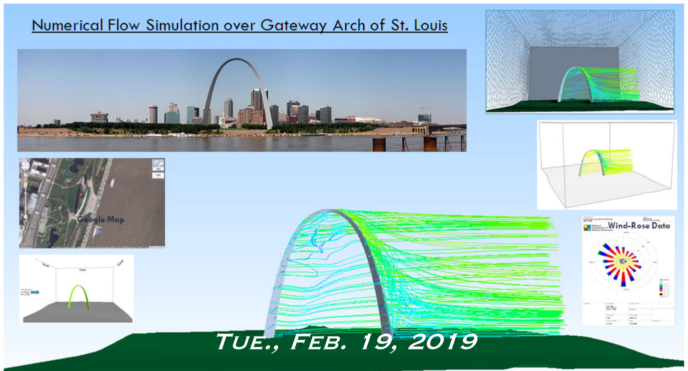
ATMS 502
CSE 566
NUMERICAL FLUID DYNAMICS



CFDmax.com

ATMS 502 - Spring 2019 2/19/19

CFDmax © 2011

ATMS 502
CSE 566

Tuesday,
19 February 2019

Class #11

Class 10 was for exam #1

Plan for Today

- 1) REVIEW
  - Smolarkiewicz, diagonal flow, splitting
- 2) NUMERICAL METHODS :
  - The origin of Lax-Wendroff
  - Fully implicit schemes
- 3) NUMERICAL METHODS :
  - Introduction to nesting

ATMS 502 - Spring 2019 2/19/19

Numerical methods: Lax-Wendroff

References:
- C001 (Lax-Wendroff)
- C006 (Finite differences)
- C007 (Taylor series)
- C052 (Advection)

ATMS 502 - Spring 2019 2/19/19

Where is 1-D Lax-Wendroff *from*?

- Lax-Wendroff a.k.a. 2nd-order Crowley
  - Taylor series expansion for ϕ^{n+1}
    - where $\phi(t+\Delta t)$ leads … with extra term on RHS

$$\phi^{n+1} = \phi^n + \Delta t \phi_t + \frac{\Delta t^2}{2}\phi_{tt} + \ldots$$

$$\textit{Substitute}: \ \phi_t = -c\phi_x, \ \phi_{tt} = c^2\phi_{xx}$$

$$\phi^{n+1} = \phi^n + \Delta t(-c\phi_x) + \frac{\Delta t^2}{2}(c^2\phi_{xx})$$

$$= \phi^n - c\Delta t\phi_x + \frac{c^2\Delta t^2}{2}\phi_{xx}$$

Using centered differences works here - this is what we are using in program #1.

ATMS 502 - Spring 2019 C001: Lax-Wendroff method 2/19/19

Review: Lax-Wendroff

- Lax-Wendroff: Taylor series, 1-way wave eqn.

must approximate these derivatives

$$\phi(t+\Delta t)=\phi(t)+\Delta t\frac{\partial\phi}{\partial t}+\frac{(\Delta t)^2}{2!}\frac{\partial^2\phi}{\partial t^2}+\ldots \text{ (change notation...)}$$

$$=\phi(t)+\Delta t\phi_t+\frac{(\Delta t)^2}{2!}\phi_{tt}+\ldots$$; neglect terms past ϕ_{tt} for Lax-W.

1-way wave equation: $\phi_t=-c\phi_x$ takes care of first time derivative.

For ϕ_{tt}, $\phi_t=-c\phi_x \Rightarrow \phi_{tt}=-c\phi_{xt}$ and:

$\phi_t=-c\phi_x \Rightarrow \phi_{xt}=-c\phi_{xx}$ so $\phi_{tt}=-c(-c\phi_{xx})=c^2\phi_{xx}$; *Thus,*

$$\phi(t+\Delta t)=\phi(t)+\Delta t(-c\phi_x)+\frac{(\Delta t)^2}{2!}(c^2\phi_{xx})$$; use centered derivatives to finish:

$$\phi(t+\Delta t)=\phi(t)-c\Delta t\left(\frac{\phi_{j+1}-\phi_{j-1}}{2\Delta x}\right)+c^2\frac{(\Delta t)^2}{2!}\left(\frac{\phi_{j+1}-2\phi_j+\phi_{j-1}}{(\Delta x)^2}\right)$$

ATMS 502 - Spring 2019 · C001: Lax-Wendroff method · 2/19/19 · 5

Notes

ATMS 502 - Spring 2019 · 2/19/19 · 6

Fully Implicit Schemes

Reference pages for this section:

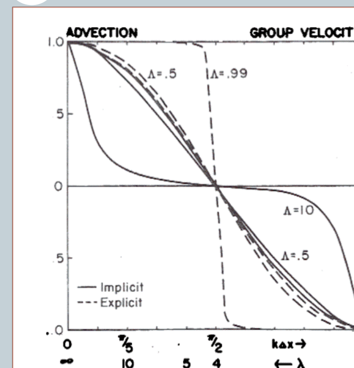
- C002 – explicit numerical methods
- C070 – implicit numerical methods

ATMS 502 - Spring 2019 · 2/19/19 · 7

Explicit vs. Implicit: advection

At right:

- Group velocities for 1-D advection for explicit (dashed) and implicit (solid)
- Explicit always more accurate for waves longer than 4Δx



Grotjahn and O'Brien, MWR **104** (1976), Fig. 6

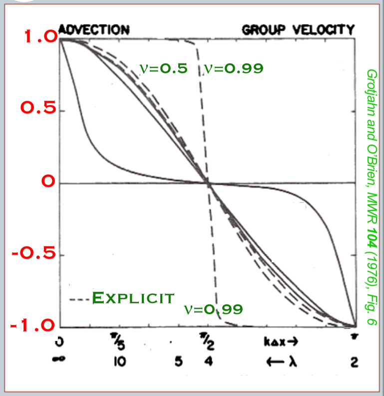
ATMS 502 - Spring 2019 · C002: Explicit methods • C046: Group velocity • C070: Implicit methods · 2/19/19 · 8

Explicit vs. Implicit: advection

9

At right:

- Group velocities for 1-D advection for explicit (dashed) and implicit (solid)
- Explicit always more accurate for waves longer than 4Δx

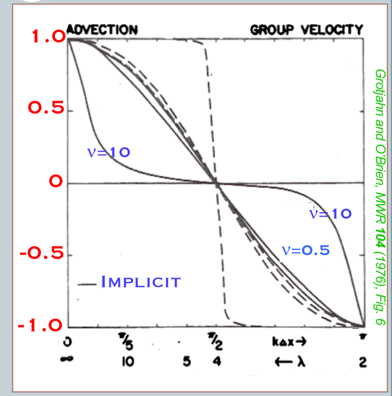


Explicit vs. Implicit: advection

10

At right:

- Group velocities for 1-D advection for explicit (dashed) and implicit (solid)
- Explicit always more accurate for waves longer than 4Δx

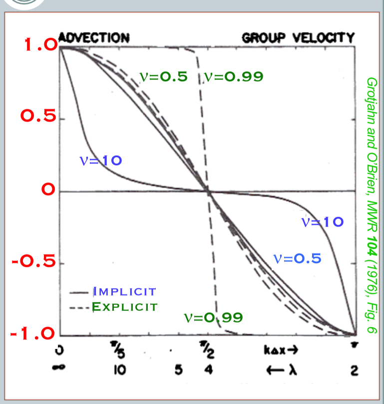


Explicit vs. Implicit: advection

11

At right:

- Group velocities for 1-D advection for explicit (dashed) and implicit (solid)
- Explicit always more accurate for waves longer than 4Δx

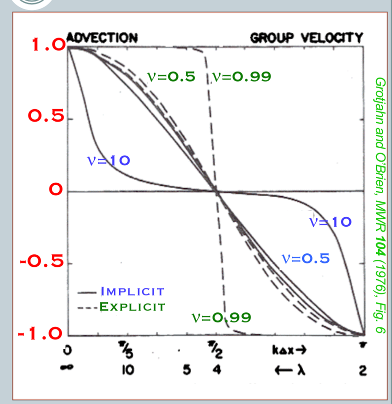


Explicit vs. Implicit: advection

12

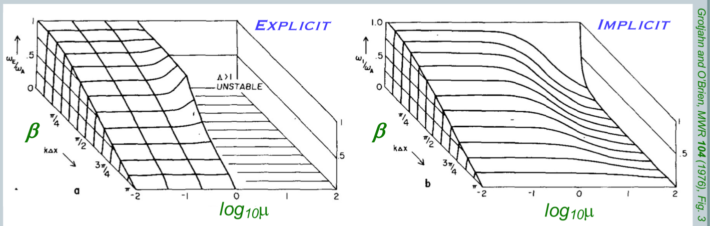
At right:

- Group velocities for 1-D advection for explicit *(dashed)* and implicit *(solid)*
- Explicit more accurate for waves longer than 4Δx
- Implicit allows large time step but has large errors, as seen here for ν=10
- Group velocities always negative for 2Δx



Explicit vs. Implicit: advection

13

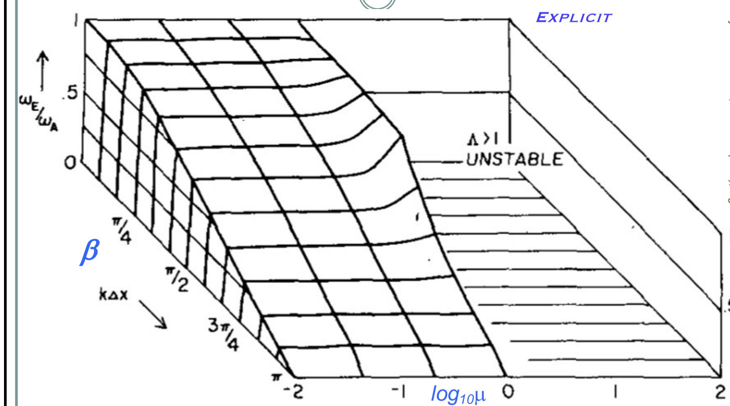


- Relative phase velocity shown in 3D for implicit and explicit schemes
- $\text{Log}_{10}(\mu)$ on abscissa; $\beta = k\Delta x$ on ordinate.
- Implicit schemes maintain stability by *slowing down the waves*.

ATMS 502 - Spring 2019 2/19/19

Explicit advection

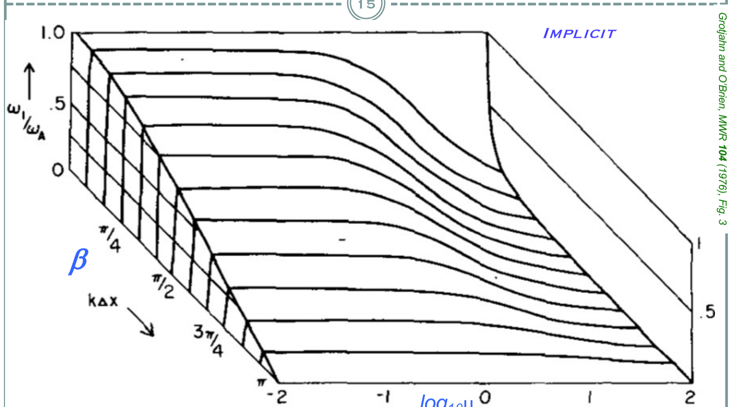
14



ATMS 502 - Spring 2019 2/19/19

Implicit advection

15



ATMS 502 - Spring 2019 2/19/19

Implicit Schemes: FTCS

16

- Recall the forward time, centered space (FTCS) scheme applied to the linear advection equation..

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta t}$$

  - ... has amplification factor: $\lambda = 1 - i\mu \sin\beta$
  - ... and is thus unstable.

- Stabilize this scheme using du/dx at *(n+1)*:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\Delta x}$$

ATMS 502 - Spring 2019 2/19/19

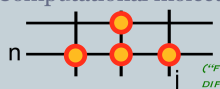
Implicit Schemes: FTCS

17

- Explicit

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x}$$

  - Computational molecule



  - Amplification factor

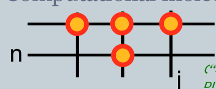
$$\lambda = 1 - i\mu\sin\beta$$

(UNSTABLE)

- Fully implicit

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -c\,\frac{u_{j+1}^{n+1}-u_{j-1}^{n+1}}{2\Delta x}$$

  - Computational molecule



  - Amplification factor

$$\lambda = \frac{1 - i\mu\sin\beta}{1+\mu^2\sin^2\beta}$$

(STABLE)

Both are O($\Delta t,\Delta x^2$) accurate. Explicit is unstable; Implicit is unconditionally stable but has large errors for large time steps, and requires matrix inversion.

ATMS 502 - Spring 2019 2/19/19

Implicit Schemes: FTCS

18

- Euler scheme:

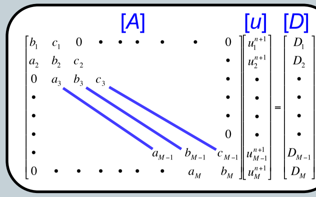
$$u_j^{n+1}-u_j^n = -(\mu/2)\left(u_{j+1}^{n+1}-u_{j-1}^{n+1}\right)$$

- Coefficients of each term:

$$\left(-\frac{\mu}{2}\right)u_{j-1}^{n+1} + (1)u_j^{n+1} + \left(\frac{\mu}{2}\right)u_{j+1}^{n+1} = u_j^n$$

$$au_{j-1}^{n+1} + bu_j^{n+1} + cu_{j+1}^{n+1} = D$$

- Solve set of equations
  - For all *j*, simultaneously
  - Matrix [A] is *tridiagonal*
    - rapid solvers exist
- Solution



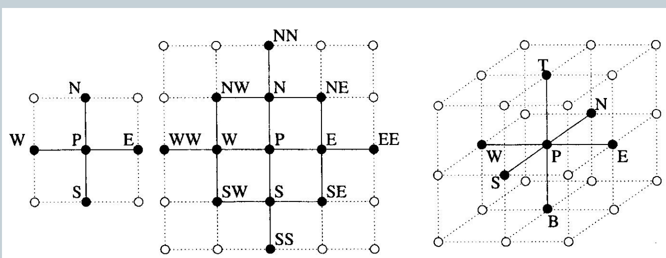
- *M* actual grid points: j=1,…,M
- Boundary values are specified at j=0 and j=M
- *M*+2 points (total)
- Initial conditions known at n=0

ATMS 502 - Spring 2019 2/19/19

Implicit schemes

19

- There are differencing schemes requiring more points - either for higher order schemes, or more dimensions.



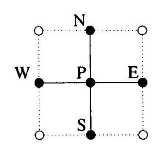
Ferziger p. 55

ATMS 502 - Spring 2019 2/19/19

Implicit schemes

20

- The matrix layout changes appropriately.



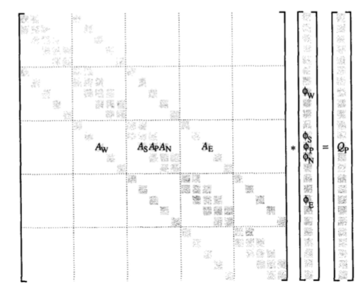


Fig. 3.5. Structure of the matrix for a five-point computational molecule (non-zero entries in the coefficient matrix on five diagonals are shaded; each horizontal set of boxes corresponds to one grid line)

Ferziger pp. 55, 57

ATMS 502 - Spring 2019 2/19/19

Nesting: introduction

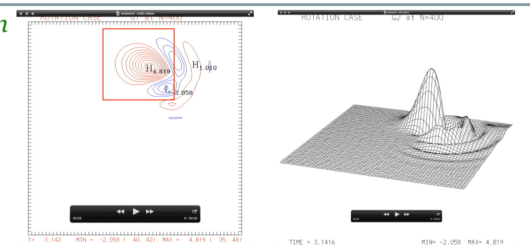
21

THE BIG PICTURE.

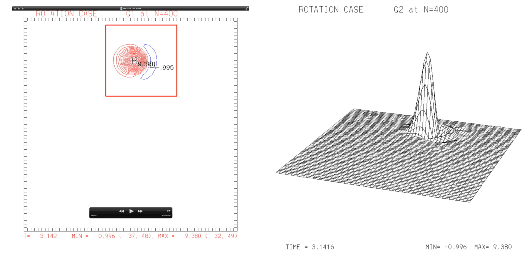
ATMS 502 - Spring 2019 | C010: Adaptive grid refinement | 2/19/19

After 1 rotation

No nest



Nest



ATMS 502 - Spring 2019 | 22 | 2/19/19

Why are we nesting

23

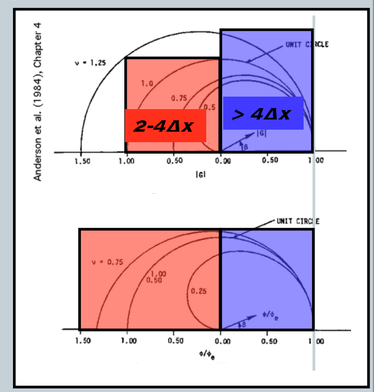
- To minimize errors -
  - Amplitude
  - Phase speed
  - Group velocity
  - Truncation error
- To better resolve small-scale features
  - ... which are damped out if represented by too few grid points (e.g. less than 4-6Δx)
- To save computation time
  - Nested grids allow higher resolution only where needed

ATMS 502 - Spring 2019 | 2/19/19

Why are we nesting

24

- To minimize errors -
  - Amplitude
  - Phase speed
  - Group velocity
  - Truncation error
- To better resolve small-scale features
- To save computation time
- Nesting: "move" solution to lower wavenumbers.



ATMS 502 - Spring 2019 | C022: Amplitude error • C023: Phase error | 2/19/19

Irregular grids

25

- Instead of regular, structured grids -

"Method of lines" - Grid points reposition themselves during integration

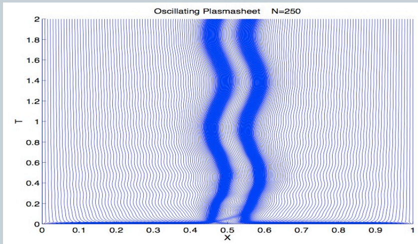


Fig. 1. The grid history in a 1D MHD simulation of an oscillating plasma sheet embedded in a vacuum. Starting with an equidistant grid of 250 grid points, the sheet boundaries are automatically recognized as regions where grid points need to be clustered. After this rapid initial adjustment (prior to times $T < 0.05$), the mesh clearly follows the oscillation.

"Dynamic grid adaptation for computational magnetohydrodynamics" - *Keppens et al., 2000*

ATMS 502 - Spring 2019 | C047: Grid refinement - Irregular grids | 2/19/19

Irregular grids

26

- Instead of regular, structured grids -

Grid distortion in response to evolving solution

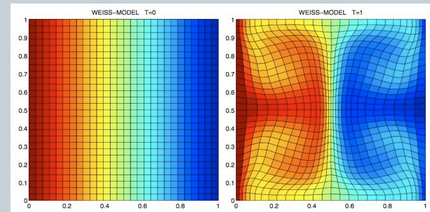


Fig. 2. A 2D kinematic flux expulsion. The left panel shows the initial cartesian mesh and the shading corresponds to the magnetic vector potential. Right panel: an imposed four-cell convection pattern causes the initially straight, uniform field to distort, which is recognized and followed by the 2D grid cell movements.

"Dynamic grid adaptation for computational magnetohydrodynamics" - *Keppens et al., 2000*

ATMS 502 - Spring 2019 | C047: Grid refinement - Irregular grids | 2/19/19

2D nesting

27

Evolving 2D grid

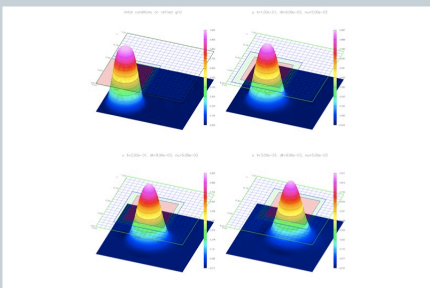


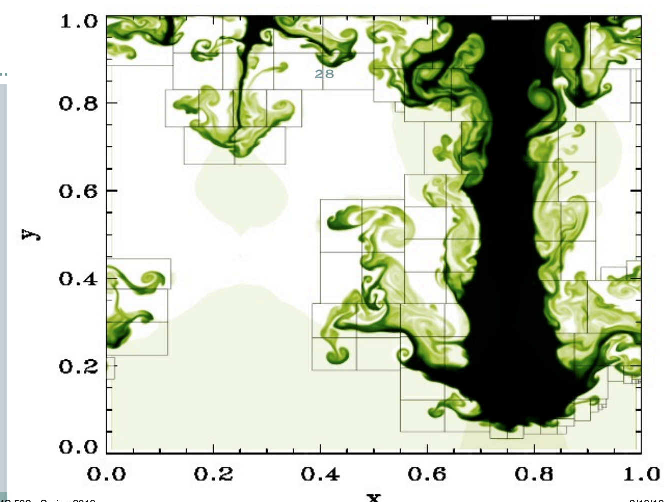
Figure 1: Results from hypeAmr, solving a convection diffusion equation with adaptive mesh refinement.

"Adaptive mesh refinement routines for Overture" - *Brown and Henshaw, 2003*

ATMS 502 - Spring 2019 | 2/19/19

"Adaptive Mesh Refinement" - *Keppens*

28

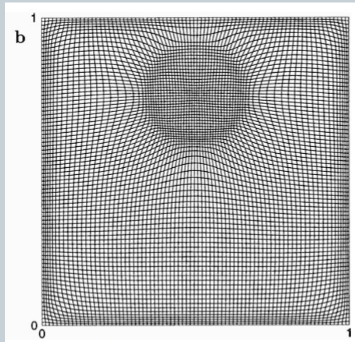


ATMS 502 - Spring 2019 | 2/19/19

Irregular grids

29

2D gridpoint redistribution



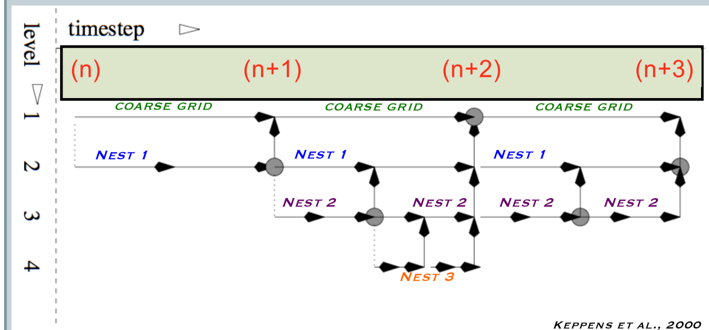
80 × 80 grid points with $\lambda_v = 4.0$, $\lambda_o = 0.0$, and $K = 4$ smoothing
(b) Improved quality grid.

"Dynamic grid adaptation using the MPDATA scheme" -
Iselin et al., 2002

ATMS 502 - Spring 2019 C047: Grid refinement - Irregular grids 2/19/19

(Time) stepping forward when nesting

30



ATMS 502 - Spring 2019 C048: Grid refinement – time integration 2/19/19

Nesting: Implementation

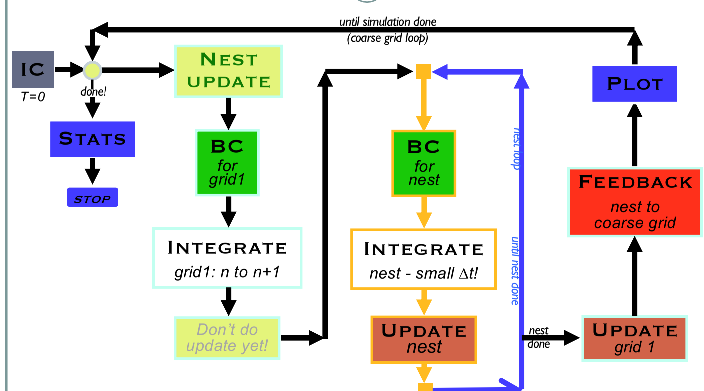
31

COMPUTER PROBLEM #4

ATMS 502 - Spring 2019 2/19/19

Program #4: main routine

32



ATMS 502 - Spring 2019 C049: Nesting – coding and sequence of operations 2/19/19