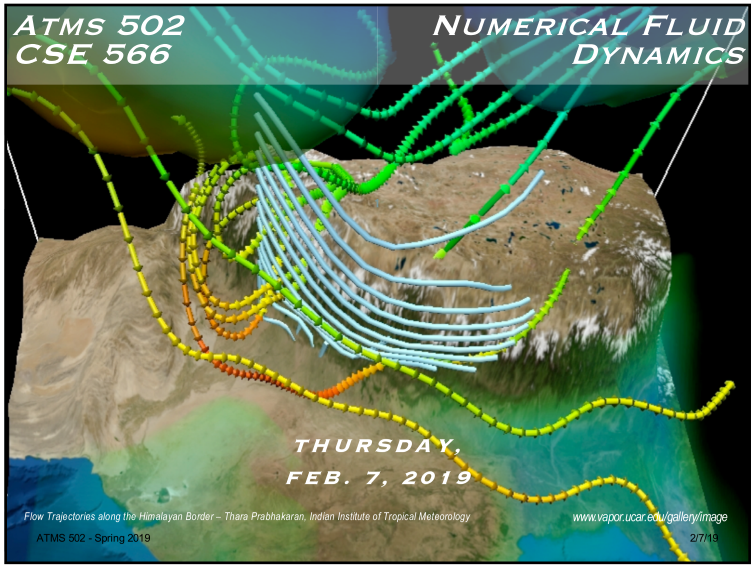

Plan for Today

ATMS 502
CSE 566

Thursday,
7 February 2019

Class #8

EXAM 1:
1 week
from
today

- 1) REVIEW
  - Stability; Phase error
- 2) CODE/DATA:
  - Program #2 - continued
- 3) NUMERICAL METHODS :
  - The modified equation
    - Shift condition
    - "Implicit" damping
    - Lax Equivalence Theorem
  - Multidimensional advection
    - *split, unsplit approaches*
    - *Smolarkiewicz discussion*

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Review: stability example #1

- We started with: $$\frac{q_j^{n+1}-q_j^n}{\Delta t}=K\frac{\left(q_{j+1}^n-2q_j^n+q_{j-1}^n\right)}{\Delta x^2}$$
- And substituted this: $$q_j^n=\tilde{q}^n e^{ikj\Delta x}$$
- Resulting in the following: $$\lambda=1-4\sigma\sin^2\left(\frac{k\Delta x}{2}\right)$$
- We looked at above expression to arrive at: $$-1\le 1-4\sigma\sin^2\left(\frac{k\Delta x}{2}\right)\le 1 \qquad 0\le\sigma\le\frac{1}{2} \qquad \Delta t\le\frac{\Delta x^2}{2K}$$

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Review: Applying von Neumann's method

Some comments on application of the Von Neumann method:

- Don't forget the absolute value …
  $|\lambda|$ must be less than or equal to 1, not just λ.
- If λ is complex, you must compute $|\lambda|^2$ …
  - for $\lambda=(x+iy)$, $|\lambda|^2 = (x^2+y^2)$
  - this must be less than or equal to 1 for stability.

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Review: Fourier series - square wave

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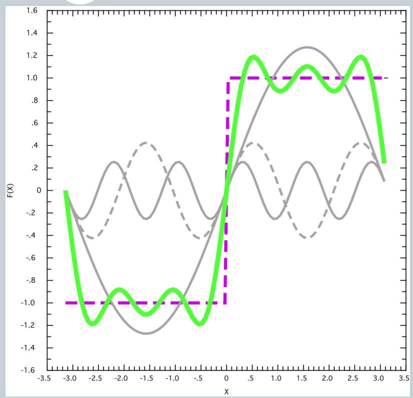
- Sum of first three waves.
- More waves …
  - Gives a better approximation
  - Gibbs oscillation remains
- Decomposition
  - … of arbitrary solution into component waves



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Review: Phase errors

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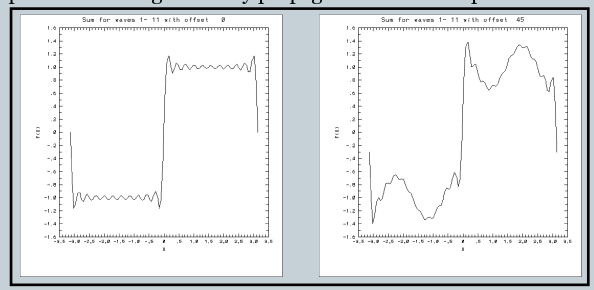
- Dispersion distorts the solution by introducing wavenumber-dependent phase errors
- The phase relationship between the different wave components is changed as they propagate at different speeds.



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Computer Program 2

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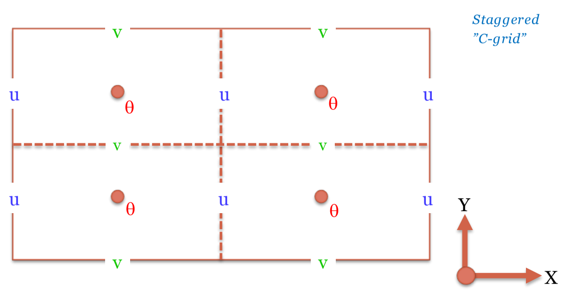

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The modified equation

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WHY WE SEE CHARACTERISTIC (OR REALLY, DOMINANT) ERRORS FOR TYPES OF SCHEMES

SEE NOTES FROM LAST CLASS!

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"Implicit" damping in weak flow

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- Knievel, Bryan, and Hacker paper (2005)
  - Weather Research and Forecasting model
  - Noticed cases with very noisy fields
  - Problem tied to flow-dependent implicit viscosity

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Divergence (contours) and terrain elevation (shaded)
Unmodified WRF Model
Domain: 3 Model level: 1
23-h forecast valid 2300 UTC 14 July 1998

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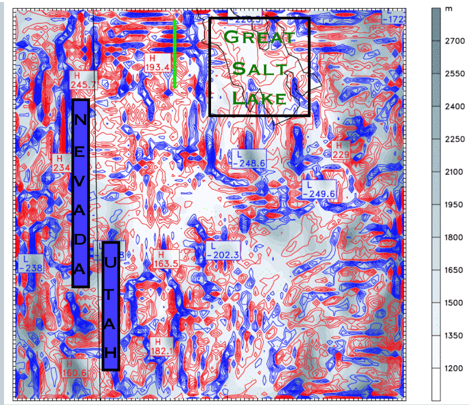

- Knievel et al. (2005)
- Figure at left: divergence (red >0; blue <0)
- Noise pronounced in areas of low wind speed.

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Summary: Errors vs. order of accuracy

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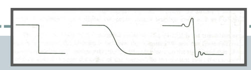

- *Explicit artificial viscosity*
  - An added term in a difference equation *designed* to add damping
- *Implicit artificial viscosity*
  - Unphysical damping as a *consequence* of the finite difference scheme
- *Summary of error properties*
  - Dispersion - distortion of waves as a result of *odd* derivative terms in the truncation error - *even* order accuracy!
  - Dissipation - reduction of gradients as a result of *even* derivative terms in the truncation error - *odd* order accuracy!
  - Diffusion: combined effect of dissipation and dispersion

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Convergence

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We've discussed:

- Finite difference approximations to derivatives
- Truncation error
  - Taylor series
  - Order of accuracy
- Consistency
- Stability
  - Von Neumann's method

Convergence -
Lax equivalence theorem

- If a finite difference scheme is:
  - Linear
  - Stable
  - Accurate of order $(\Delta t)^p$, $(\Delta x)^q$, then
- It is *convergent* of order (p,q)
  - *This has not been shown to apply to nonlinear PDE's !!*
  - *Durran, §2.1.3, p. 40*

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Multidimensional advection

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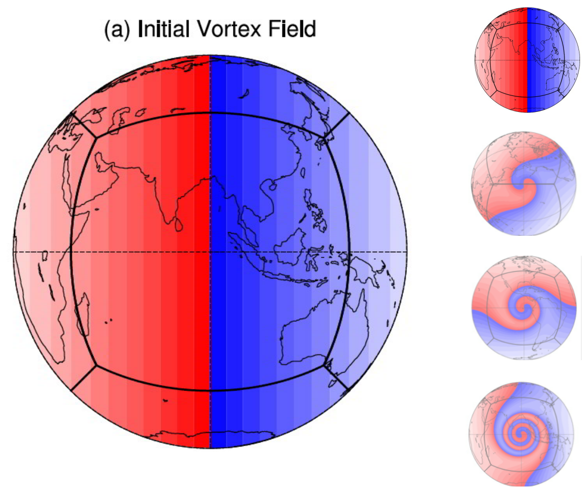
MULTIDIMENSIONAL ADVECTION AND DIRECTIONAL SPLITTING – ISSUES AND SOLUTIONS.

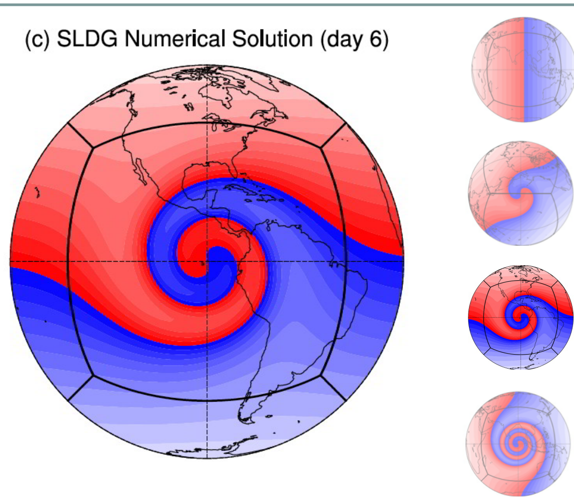
ATMS 502 - Spring 2019 C052: Advection 2/7/19

(a) Initial Vortex Field



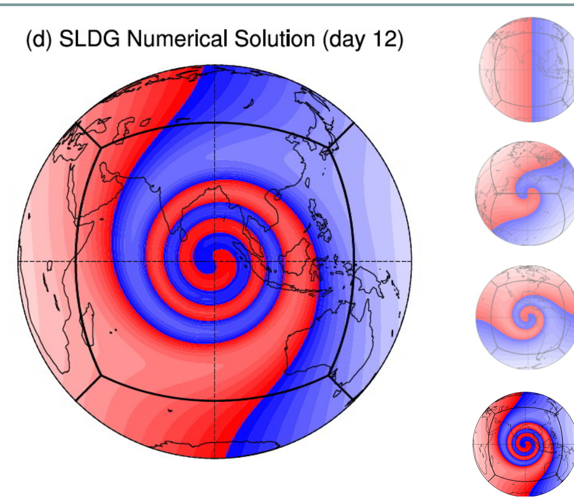
Guo, Nair, Qiu, 2013: *A conservative semi-Lagrangian discontinuous Galerkin scheme on the cubed-sphere*

(c) SLDG Numerical Solution (day 6)



Guo, Nair, Qiu, 2013: *A conservative semi-Lagrangian discontinuous Galerkin scheme on the cubed-sphere*

(d) SLDG Numerical Solution (day 12)



Guo, Nair, Qiu, 2013: *A conservative semi-Lagrangian discontinuous Galerkin scheme on the cubed-sphere*

Multidimensional advection

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We're already doing this in our programs!

- But let's look in detail at some *issues* when moving from 1-D to higher dimensions:
  - can we simply add more terms for 2-D and 3-D?
    - methodology
    - stability
    - consistency
    - what about directional splitting?

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1-D to 2-D: Options

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- Add terms to difference expression
  - this is an *unsplit* approach
    - unsplit in the sense that each operator is independent.
    - discussion
- Directional splitting
  - same operator, applied in multiple dimensions
    - this is a *split* approach, IF we use results of each step in next
    - discussion

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Extending Lax-Wendroff, part 1

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- 1-D Lax-W: $\delta_t \phi = -c\delta_{2x}\phi_j^n + \frac{c^2\Delta t}{2}\delta_{xx}\phi_j^n$
  - In Durran's operator notation

$$\phi^{n+1} = \left[1 - L(\Delta t)\right]\phi^n \text{ where } L(\Delta t) = c\Delta t\delta_{2x} - \frac{c^2\Delta t^2}{2}\delta_{xx}$$

  - Let's try a simple ***unsplit*** extension to 2-D.

$$\phi_{i,j}^{n+1} = (1 - L_1 - L_2)\phi_{i,j}^n \quad \begin{cases} L_1 = U\Delta t\delta_{2x} - \frac{U^2\Delta t^2}{2}\delta_{xx} \\ L_2 = V\Delta t\delta_{2y} - \frac{V^2\Delta t^2}{2}\delta_{yy} \end{cases}$$

L_1, L_2 = x and y advection!

ATMS 502 - Spring 2019 C001: Lax-Wendroff method • C052: Advection 2/7/19

Extending Lax-Wendroff, part 1

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- Von Neumann analysis:

$$\lambda = 1 - \frac{U\Delta t}{2\Delta x}2i\sin k\Delta x + \frac{U^2\Delta t^2}{2\Delta x^2}(2\cos k\Delta x - 2) - \frac{V\Delta t}{2\Delta y}2i\sin l\Delta y + \frac{V^2\Delta t^2}{2\Delta y^2}(2\cos l\Delta y - 2)$$

$$\lambda = 1 - i\mu\sin k\Delta x - 2\mu^2\sin^2\frac{k\Delta x}{2} - i\nu\sin l\Delta y - 2\nu^2\sin^2\frac{l\Delta y}{2}$$

  - Let $\mu=\nu$ and $k\Delta x=l\Delta y=\pi/3$:

$$k\Delta x = l\Delta y = \pi/3 \;\Rightarrow\; \sin() = \sqrt{3}/2; \quad |\lambda|^2 = 1 + \mu^2 + \mu^4 > 1$$

  - *This approach is unstable.*
    - simply adding terms for extra dimensions did not work.

ANALYSIS FOLLOWING WILHELMSON

ATMS 502 - Spring 2019 C020: von Neumann stability condition 2/7/19

Extending Lax-Wendroff, part 2

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- 1-D Lax-W: $\delta_t\phi = -c\delta_{2x}\phi_j^n + \frac{c^2\Delta t}{2}\delta_{xx}\phi_j^n$
  - How about our directional *splitting* approach?

$$\phi_{i,j}^* = [1 - L_1(\Delta t)]\phi_{i,j}^n \qquad \phi_{i,j}^{n+1} = [1 - L_2(\Delta t)]\phi_{i,j}^* \quad \textit{where} \quad \begin{cases} L_1 = U\Delta t\delta_{2x} - \frac{U^2\Delta t^2}{2}\delta_{xx} \\ L_2 = V\Delta t\delta_{2y} - \frac{V^2\Delta t^2}{2}\delta_{yy} \end{cases}$$

  - If $|A_1| \le 1$ and $|A_2| \le 1$, the scheme is stable.
  - This **2-D** stability condition is the **same as** for **1-D**; is *less restrictive* than many 2-D schemes

Why did the unsplit attempt fail?

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- Crowley extension (Smolarkiewicz 1982):
  - Failed because cross-derivative terms **were missing**
    - *noted by Leith (1965); see Smolarkiewicz p. 1969, right column*
  - Smolarkiewicz (1982):
    - a two-dimensional **second-order** Taylor-series yields the following, with a new added term*:

$$\phi_{ij}^{n+1} = \phi_{ij}^{n} - u\Delta t\delta_{2x}\phi - v\Delta t\delta_{2y}\phi + \frac{u^2\Delta t^2}{2}\delta_{xx}\phi + \frac{v^2\Delta t^2}{2}\delta_{yy}\phi + uv\Delta t^2\delta_{xy}\phi$$

  - Stability condition: $\sqrt{(u\Delta t/\Delta x)^2 + (v\Delta t/\Delta y)^2} \le 0.5$
  - Order ($\Delta t + \Delta x^2 + \Delta y^2$)

*he also derives a 3d form.

Smolarkiewicz (1982)

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- Crowley extension (cont)
  - Smolarkiewicz also found:
    - Diagonal flow ...
      - with $u\Delta t/\Delta x = v\Delta t/\Delta y$
      - is the case of maximum instability;
    - Most unstable waves are those with $k\Delta x = l\Delta y$
    - Including the "cross term" *not enough* for stability with

$$\sqrt{(u\Delta t/\Delta x)^2 + (v\Delta t/\Delta y)^2} \approx 1$$

    - He went on to derive a scheme with higher-order approximations and averaging
      - this allowed a larger time step.

Directional splitting ('fractional steps')

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- What about directional splitting?
  - it works; contains cross-term; is efficient ... but ...
- Substitute..

$$\phi^* = \phi^n - \Delta t U\delta_x\phi^n; \quad \phi^{n+1} = \phi^* - \Delta t V\delta_y\phi^*$$
$$\phi^{n+1} = \phi^* - \Delta t V\delta_y\phi^*$$
$$= \phi^n - \Delta t U\delta_x\phi^n - \Delta t V\delta_y\phi^n + \Delta t^2 UV\delta_{xy}\phi^n + \ldots ?$$

- What term are we missing in this equation?
  - Compare to the fully multidimensional scheme from Smolarkiewicz, this term does not belong here.
  - This "source" term is a consequence of the splitting.

ANALYSIS FOLLOWING XUE (2000)