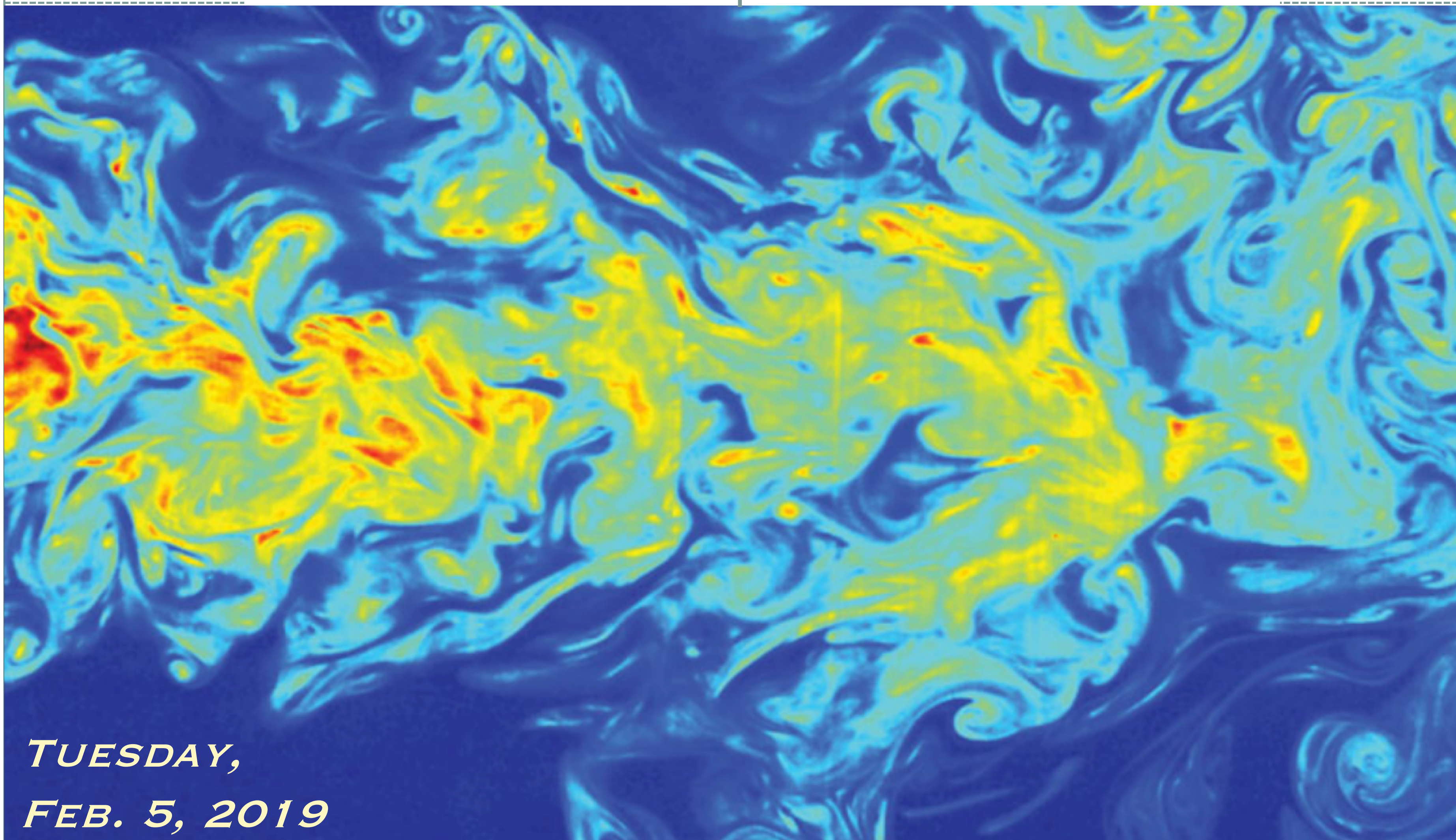
Atms 502, CSE 566

Numerical Fluid Dynamics



TUESDAY,
FEB. 5, 2019

"Momentum and scalar transport at the turbulent/non-turbulent interface of a jet" - Westerweel, Fukushima, Pedersen and Hunt; link

ATMS 502
CSE 566

Tuesday,
5 February 2019

Class #7

Homework #1
prob #4 typo:
should be
$2\Delta x$ not Δx !!

Plan for Today

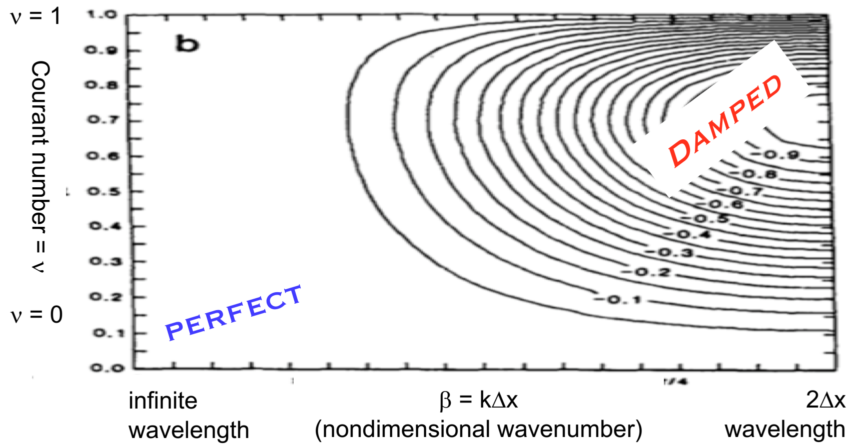
- 1) REVIEW
  Takacs vs. Polar plots
- 2) NUMERICAL METHODS :
  - Stability, continued:
    - Norms; von Neumann's method
    - Apply to a numerical method
    - Operator definitions
  - Phase error
    - *wavelength-dependent phase speeds*
  - The modified equation
- 3) CODE/DATA:
  Program #2 - continued

Review: Plots of scheme behavior

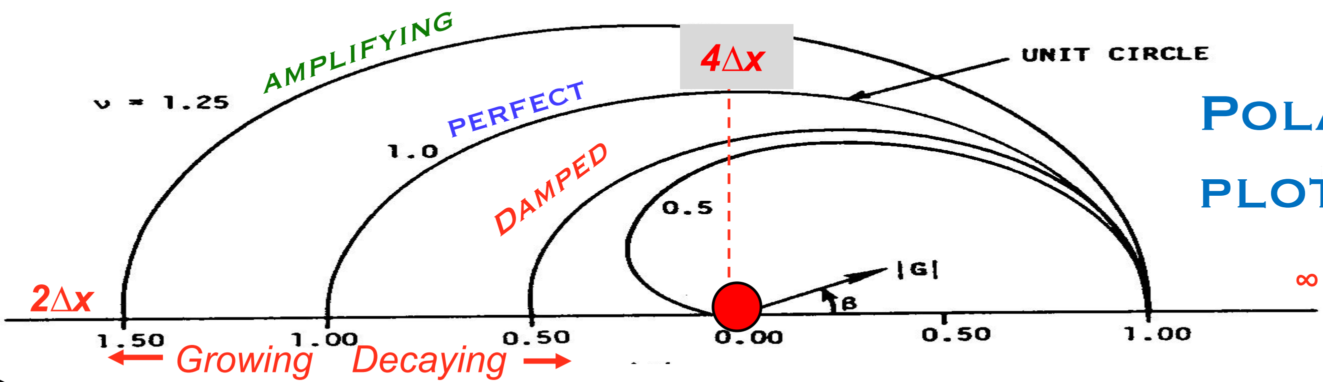
Amplitude

Takacs' plots

(not showing the same numerical method)



Polar plots



Anderson et al., chapter 4

Stability

OBJECTIVES:

DEVELOP THEORY AND METHODOLOGY FOR DETERMINING IF, HOW, AND WHEN A SCHEME HAS SATISFACTORY STABILITY.

FOLLOWING NOTES HANDED OUT IN LAST CLASS !!

References:
- A009 – Instability (physical)
- C015 – Instability (numerical)

Operator definitions

USED FOR REMAINDER OF CLASS

FOLLOWING NOTES HANDED OUT IN LAST CLASS !!

Dispersion & phase error

IN CONTEXT OF: FOURIER SERIES

References:
- C016 – Fourier series
- C023 – Dispersion
- C033 – Gibbs phenomenon

Fourier series

- Assuming we have a *periodic* function of period 2π, we can represent it with the following trig series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx \; + \; b_n \sin nx \right)$$

- Solve using the *Euler Formulas*:

Period=T

$$\left.\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{aligned}\right\} \quad n = 1,2,\ldots$$

OR

$$\left.\begin{aligned} a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} f(t)\,dt \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,dt \\ b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,dt \end{aligned}\right\} \quad n = 1,2,\ldots$$

Fourier series

- Let's look at the representation of a square wave.

$$f(x) = \begin{cases} -k & \text{when } -\pi < x < 0 \\ k & \text{when } 0 < x < \pi \end{cases}$$

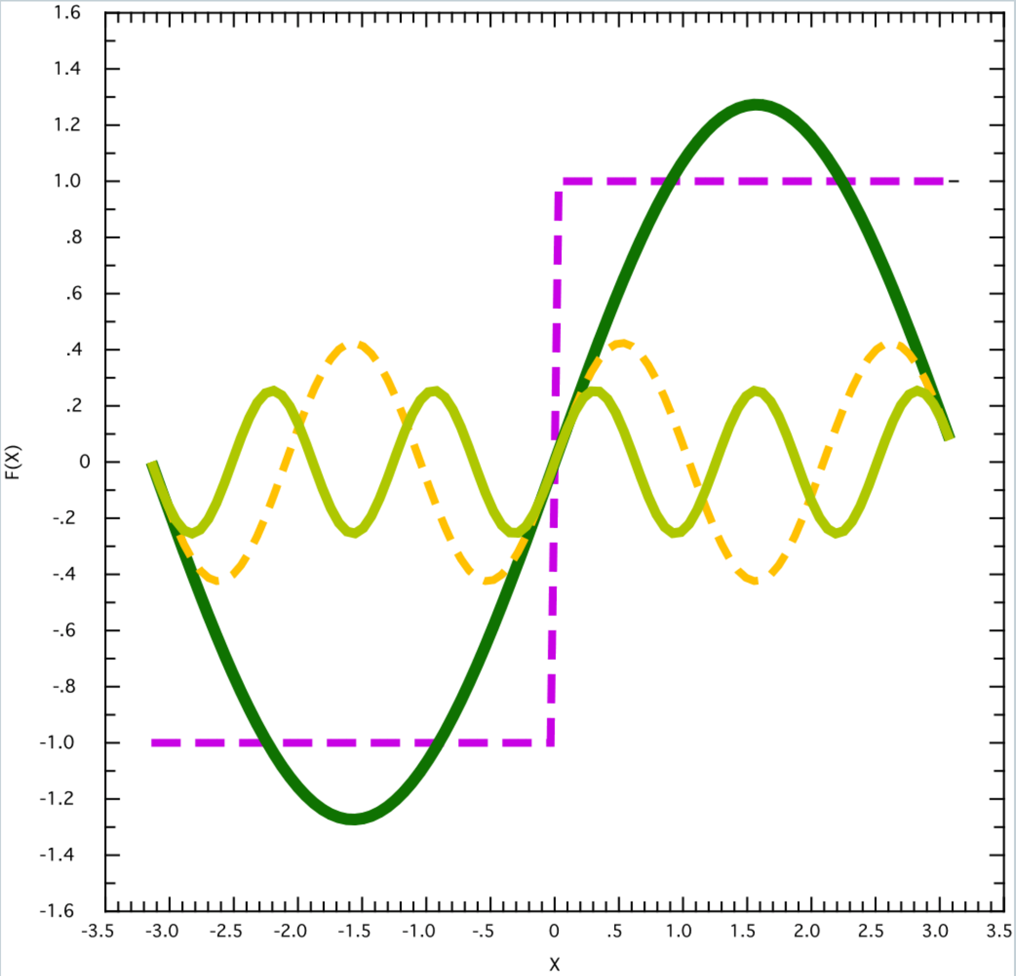
- The analytical solution turns out to be:

$$\left.\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{4k}{n\pi} \end{aligned}\right\}, \text{ n} = 1,3,5\ldots \text{ such that } f(x) \approx \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right)$$

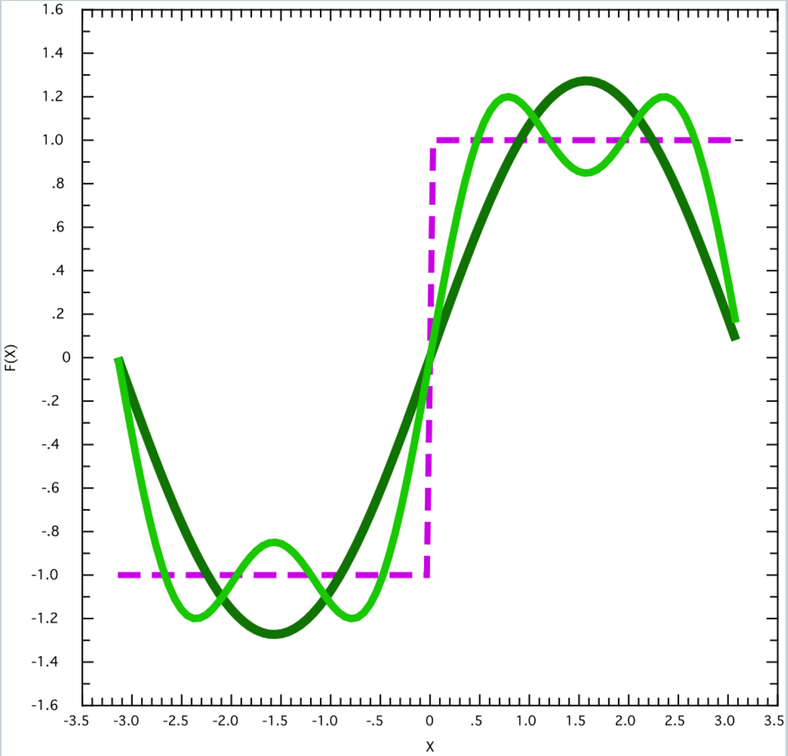
Fourier series & square wave

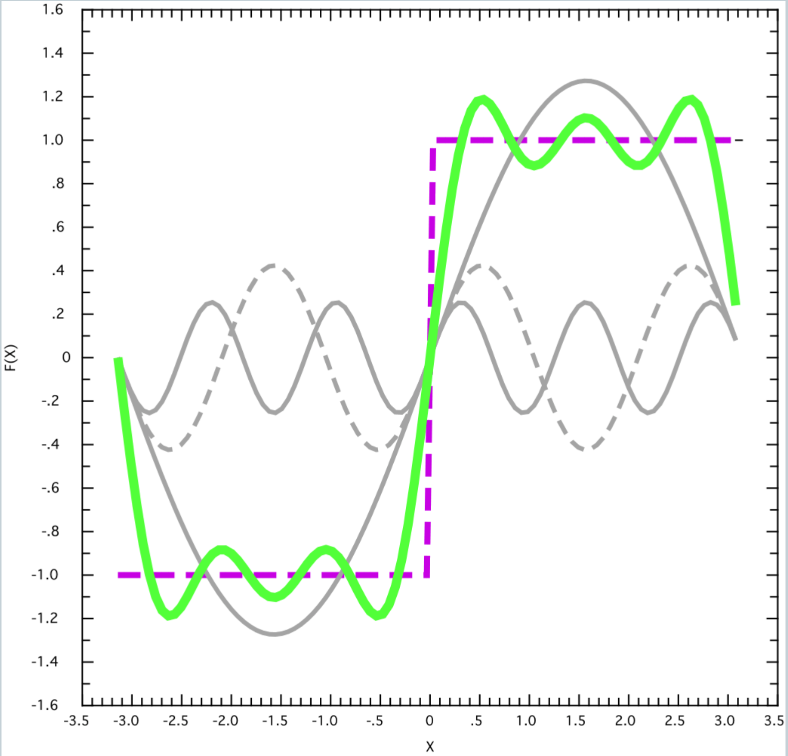
- Fourier series *cannot be used (with the strict equality) for a discontinuity; doing so invokes the **Gibbs phenomenon***.
- "While the rms error of the Fourier series goes to zero for an infinite number of terms, equality at every point is not guaranteed; the Gibbs phenomenon peaks have finite height and zero width."
- Coefficients in a truncated Fourier series can be used to reduce or eliminate the Gibbs effect; see References for more on these windowing methods.

- N=1,3,5: three waves shown: sin(x), sin(3x), sin(5x) …

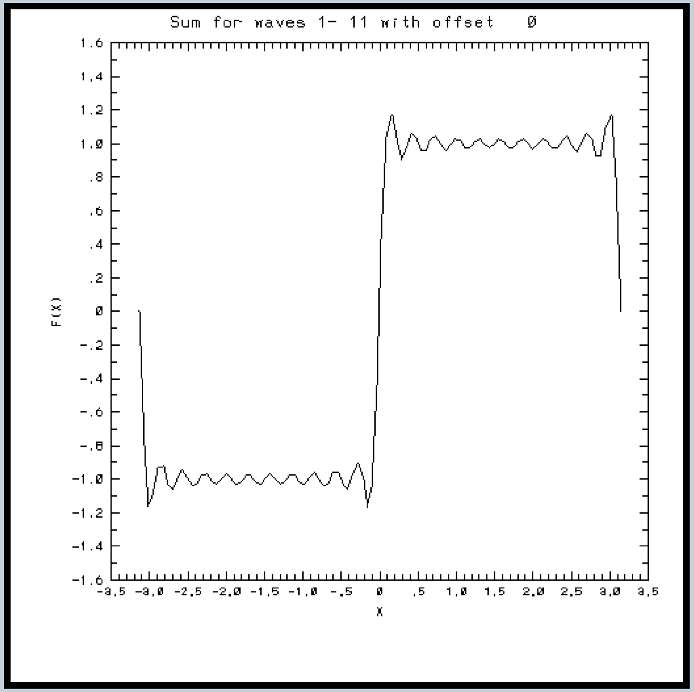


- Using one wave vs. sum of two waves
- Sum of first three waves.
- A better approximation.





What if our scheme has phase errors?

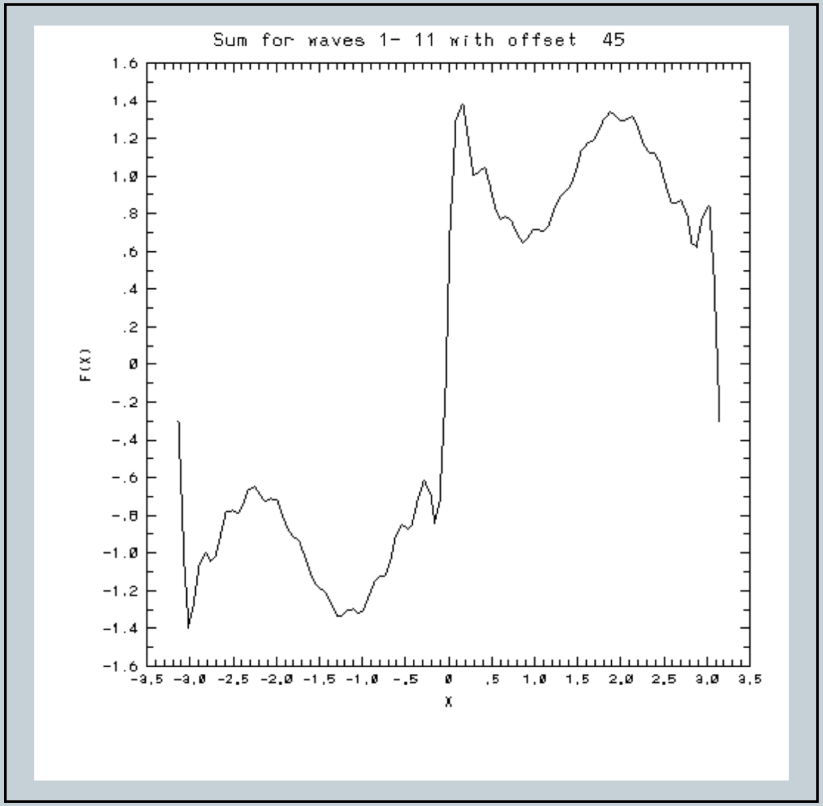


- *In this example, there is no phase shift - we add the terms correctly.*

- We'll take *one component - 1 harmonic* - and change it
- The second wave -- with *wave number k=3* -- will have a phase shift added to it.
- There are 11 contributing waves; all other components - the other 10 waves - will remain the same.

Phase errors

- Dispersion distorts the solution by introducing wavenumber-dependent phase errors
- The phase relationship between the different wave components is changed.
- In this example, the 2nd wave component was shifted by +45°



The modified equation

WHY WE SEE CHARACTERISTIC
(OR REALLY, DOMINANT)
ERRORS FOR TYPES OF SCHEMES

The modified equation

- Upstream scheme - truncation error:

$$u_t + cu_x = -\frac{\Delta t}{2}u_{tt} + \frac{c\Delta x}{2}u_{xx} \;+ higher\ order\ terms \qquad (1)$$

- To say more about this scheme, we want the right side to be in terms of X derivatives, *only*.
  - Take equation (1), and take d/dt:

$$u_{tt} + cu_{xt} = -\frac{\Delta t}{2}u_{ttt} + \frac{c\Delta x}{2}u_{xxt} \;+ higher\ order\ terms \qquad (2)$$

  - And now take -c•d/dx of (1):

$$-cu_{tx} - c^2u_{xx} = \frac{c\Delta t}{2}u_{ttx} - \frac{c^2\Delta x}{2}u_{xxx} \;+ higher\ order\ terms \qquad (3)$$

Now: add (2) + (3) … for u_{tt} = …

Modified equation: implicit viscosity

- Our original equation:

$$u_t + cu_x = -\frac{\Delta t}{2} u_{tt} + \frac{c\Delta x}{2} u_{xx} + \textit{higher order terms} \quad (1)$$

- Our expression for u_{tt} was:

$$u_{tt} = c^2 u_{xx} + \Delta t\left(-\frac{u_{ttt}}{2} + \frac{c}{2} u_{ttx} + \ldots\right) + \Delta x\left(\frac{c}{2} u_{xxt} - \frac{c^2}{2} u_{xxx} + \ldots\right) \quad (4 = (\text{eq. } 2) + (3))$$

- Substituting, we get:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} + (..)(\Delta x)^2 u_{xxx} + \textit{higher order terms}$$

  - This is the *modified equation*
  - It is what is *actually solved* by the F.D. method
  - U_{xx} term: "*implicit*" *artificial viscosity!* *(not an implicit numerical method!)*

Modified equation: dissipative errors

- Modified equation:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} + (..)(\Delta x)^2 u_{xxx} + higher\ order\ terms$$

- Why is this diffusive?
  - Recap: we started with a *hyperbolic* PDE, $u_t+cu_x=0$.
  - The modified equation tells us what we ***really*** solving.
  - Now our analysis reveals terms like: $u_t=()\cdot u_{xx}$
    - This is a parabolic equation!
      - wait, weren't we solving a *hyperbolic (transport)* equation?
    - Parabolic > for example, heat transfer
      - Bottom line: Dissipation!

Diffusive upstream method: polar plot

- Modified equation for the upstream method:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} + (..)(\Delta x)^2 u_{xxx} + higher\ order\ terms$$

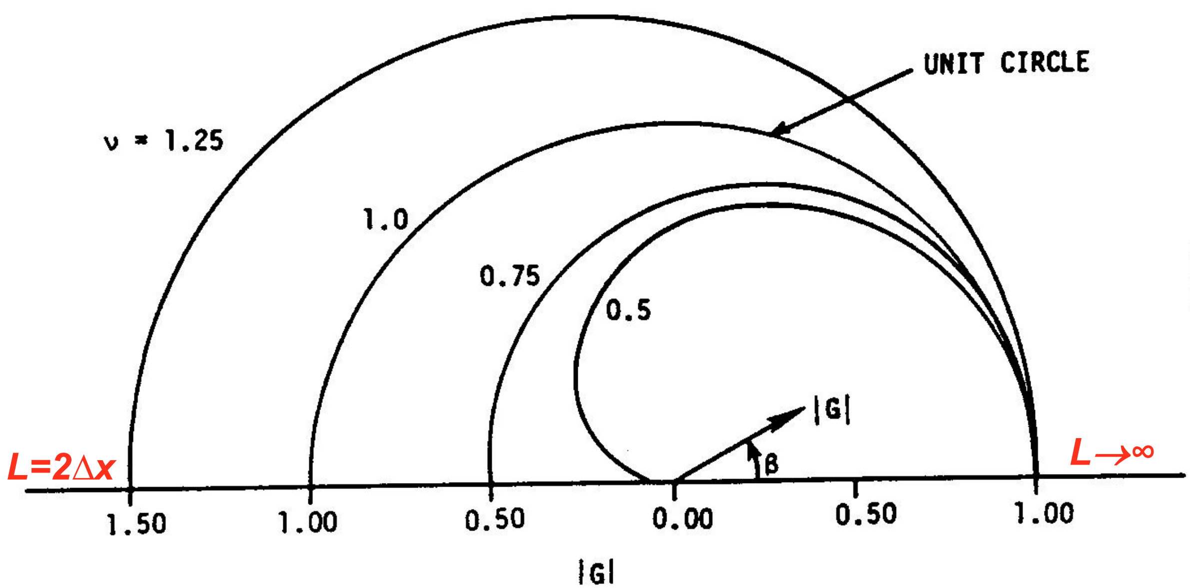


Figure 4-2 Amplification factor modulus for upstream differencing scheme.

Shift condition

- Modified equation for the upstream method:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} + (..)(\Delta x)^2 u_{xxx} + higher\ order\ terms$$

- What if the Courant number ν=1 here?
  - The u_{xx} term on the right side of the modified equation disappears
  - In fact, we know this scheme has a *shift condition* when the Courant number ν=1.
    - this means the solution is shifted one grid point per time step.

$$u_j^{n+1} = u_j^n - \nu\left(u_j^n - u_{j-1}^n\right) \;\; ; \;\; \nu = \left(\frac{c\Delta t}{\Delta x}\right)$$

Problems with upwinding-type schemes

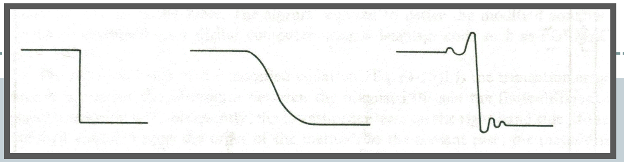
- Note the problem with upstream-type methods:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\nu)u_{xx} + (..)(\Delta x)^2 u_{xxx} + \textit{higher order terms}$$

- The implicit diffusion u_{xx} from this scheme is dependent on the local flow speed.
  - So you have uneven damping throughout your flow
  - Some modelers rely on this *implicit** damping to stabilize their solution. We'll return to this later.

*Again, *implicit* here refers to the damping, not an implicit numerical method.

Summary: Errors vs. order of accuracy



- *Explicit artificial viscosity*
  - An added term in a difference equation *designed* to add damping
- *Implicit artificial viscosity*
  - Unphysical damping as a *consequence* of the finite difference scheme
- *Summary of error properties*
  - Dispersion - distortion of waves as a result of *odd* derivative terms in the truncation error - *even* order accuracy!
  - Dissipation - reduction of gradients as a result of *even* derivative terms in the truncation error - *odd* order accuracy!
  - Diffusion: combined effect of dissipation and dispersion

Convergence

We've discussed:

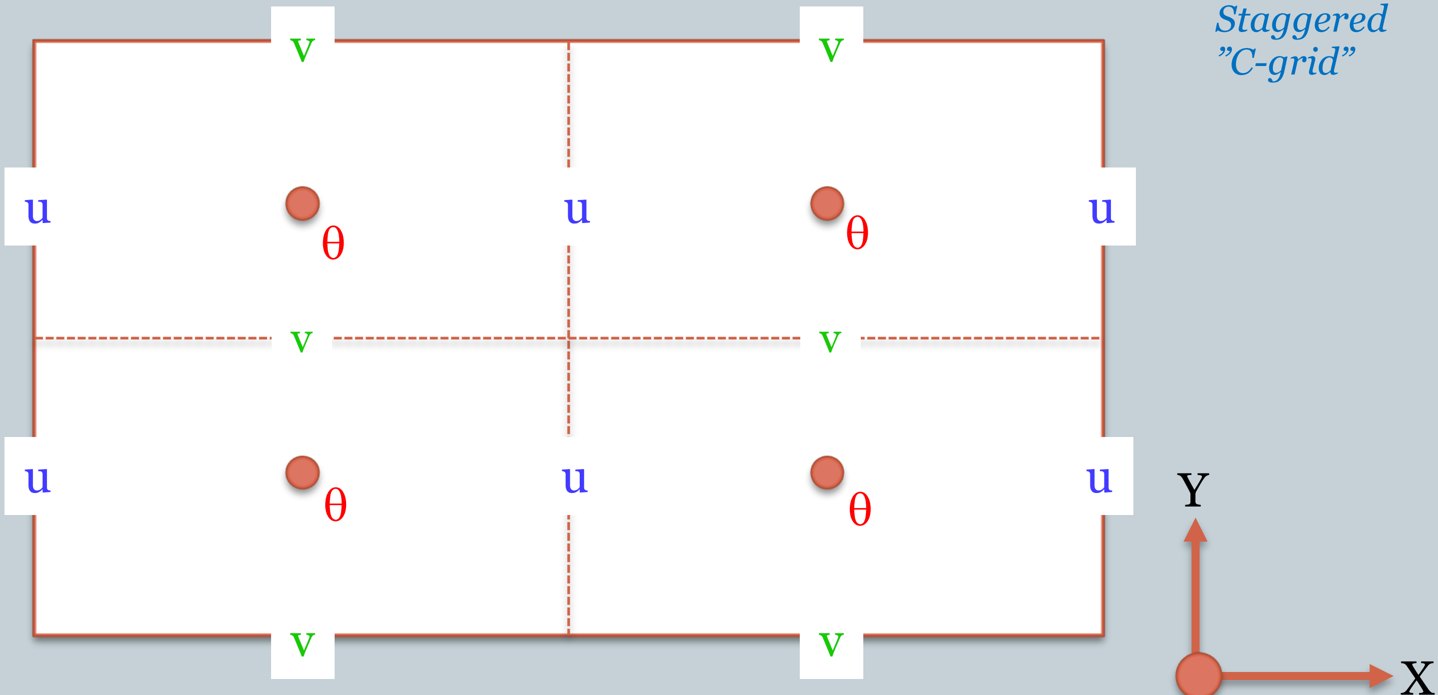
- Finite difference approximations to derivatives
- Truncation error
  - Taylor series
  - Order of accuracy
- Consistency
- Stability
  - Von Neumann's method
  - CFL

Convergence -

Lax equivalence theorem

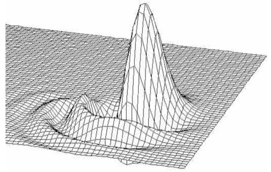
- If a finite difference scheme is:
  - Linear
  - Stable
  - Accurate of order $(\Delta t)^p$, $(\Delta x)^q$, then
- It is convergent of order (p,q)
  - *This has not been shown to apply to nonlinear PDE's !!*
  - *Durran, §2.1.3, p. 40*

Computer Program 2



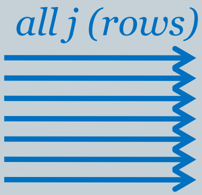
Staggered "C-grid"

Program 2: Advection

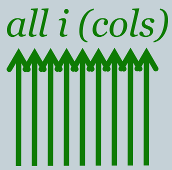


- **Advection**
  - I set up 1-D arrays in my advection routines –
    - q1d(0:nx+1), u1d(nx+1), v1d(ny+1) *no ghost points for U, V !!*

- Advecting rows (X)
  - copy q1(i,j) to q1d
  - copy u(i,j) to u1d
  - pass q1d, u1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)



- Advecting columns (Y)
  - copy q1(i,j) to q1d
  - copy v(i,j) to v1d
  - pass q1d, v1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)



discuss: how many 2D q() arrays here?