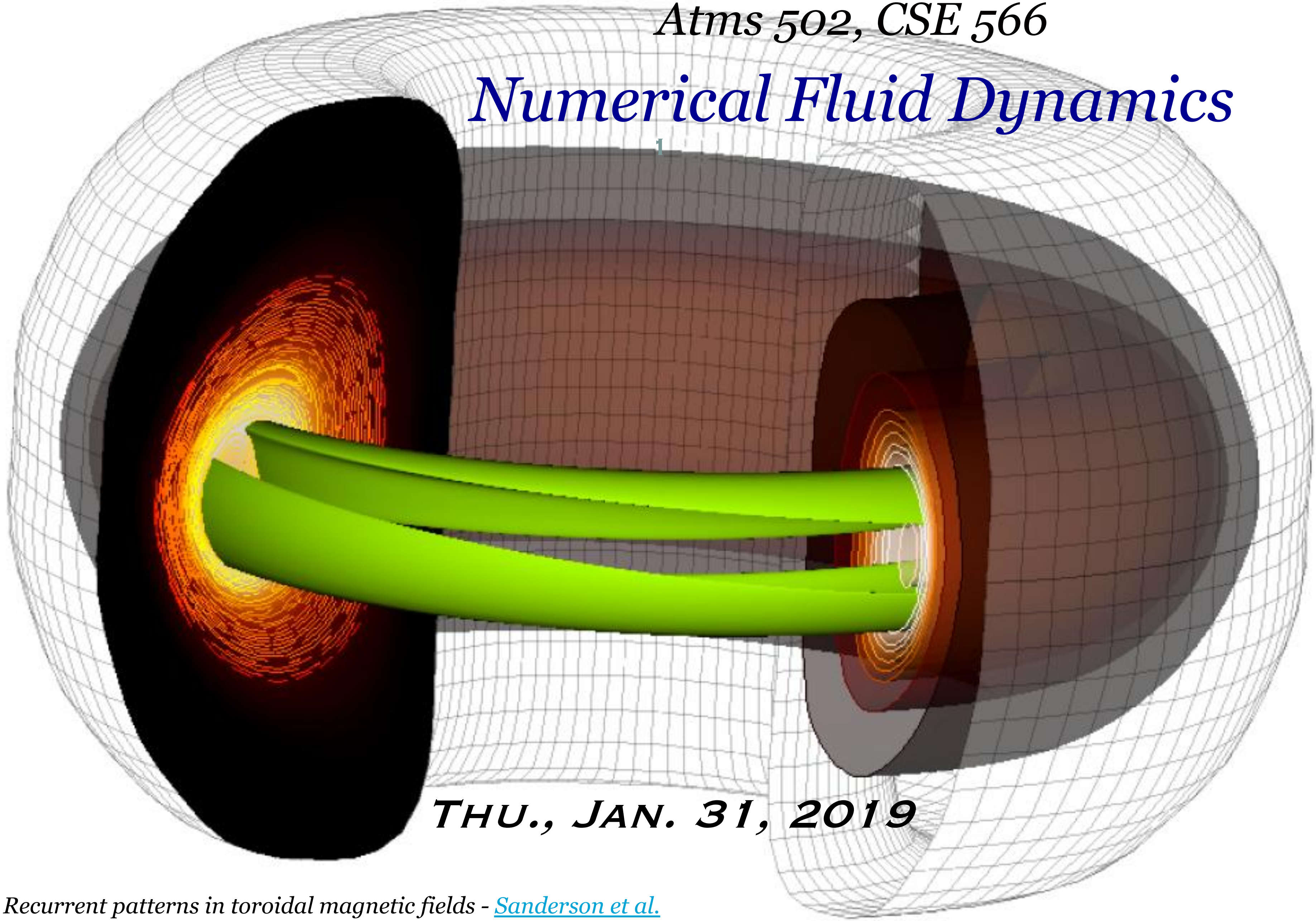
Atms 502, CSE 566

Numerical Fluid Dynamics



THU., JAN. 31, 2019

Recurrent patterns in toroidal magnetic fields - [Sanderson et al.]

ATMS 502
CSE 566

Thursday,
31 January 2019

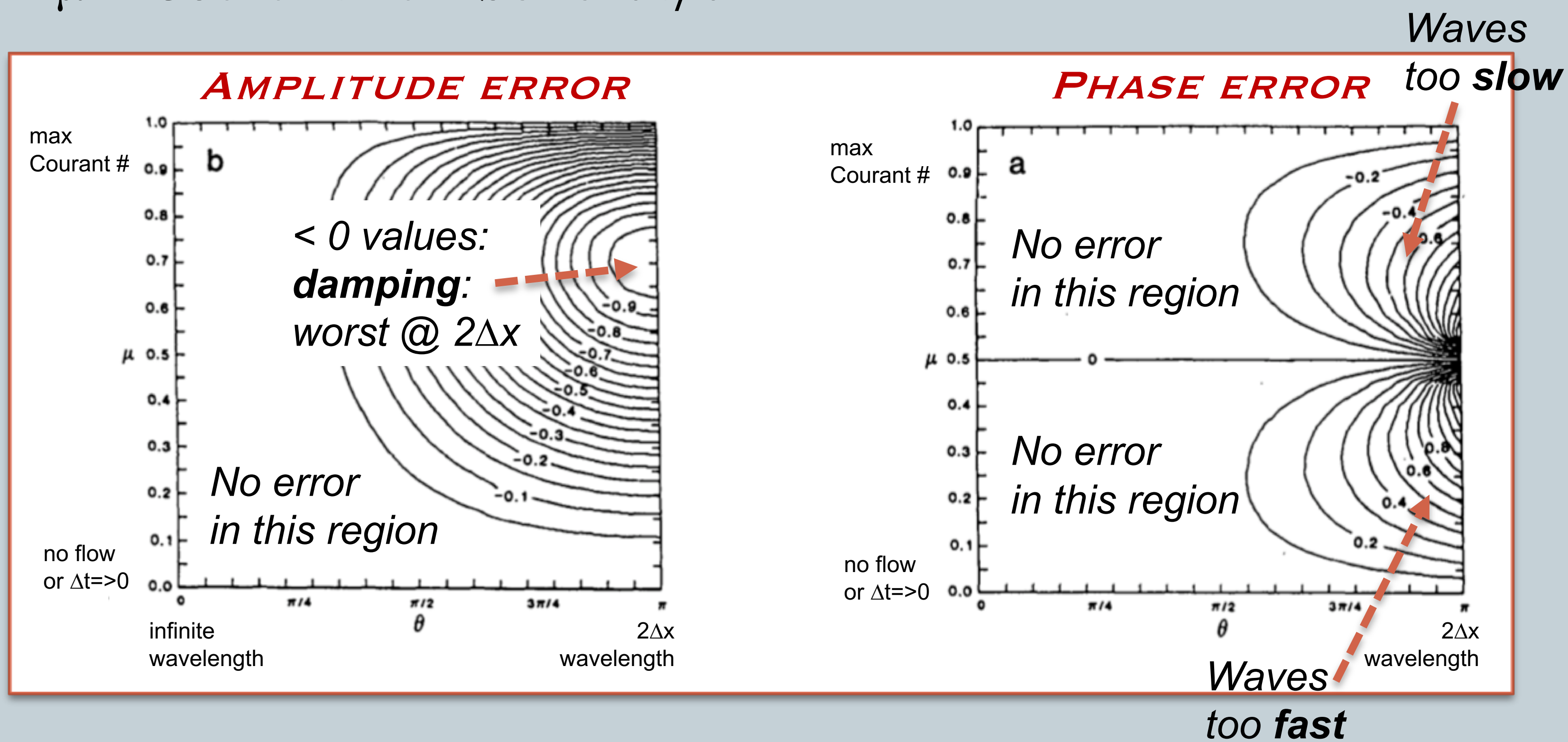
Class #6

Plan for Today

- 1) REVIEW
  Takacs plots, method, error calc.
- 2) CODE/DATA:
  Program #2 - handout
- 3) NUMERICAL METHODS :
  Polar plots
  Stability
  Consider one harmonic …
  Norms
  von Neumann's method
  Apply to a numerical method
  Operator definitions

Review: Takacs' plots

- Plots – Amplitude and Phase Error
  - θ = kΔx = nondimensional wavenumber
  - μ = Courant number c*dt/dx



Review: Takacs' scheme

- Goal: balance dissipation, dispersion *(eqn 4.3)*
  - Chooses 2-step scheme for simplicity, cost
  - His method is 2nd order + "some of improved phase characteristics associated w/third-order scheme"

$$q_j^{n+1} = a_1 q_{j+1}^n + a_0 q_j^n + a_{-1} q_{j-1}^n + a_{-2} q_{j-2}^n$$

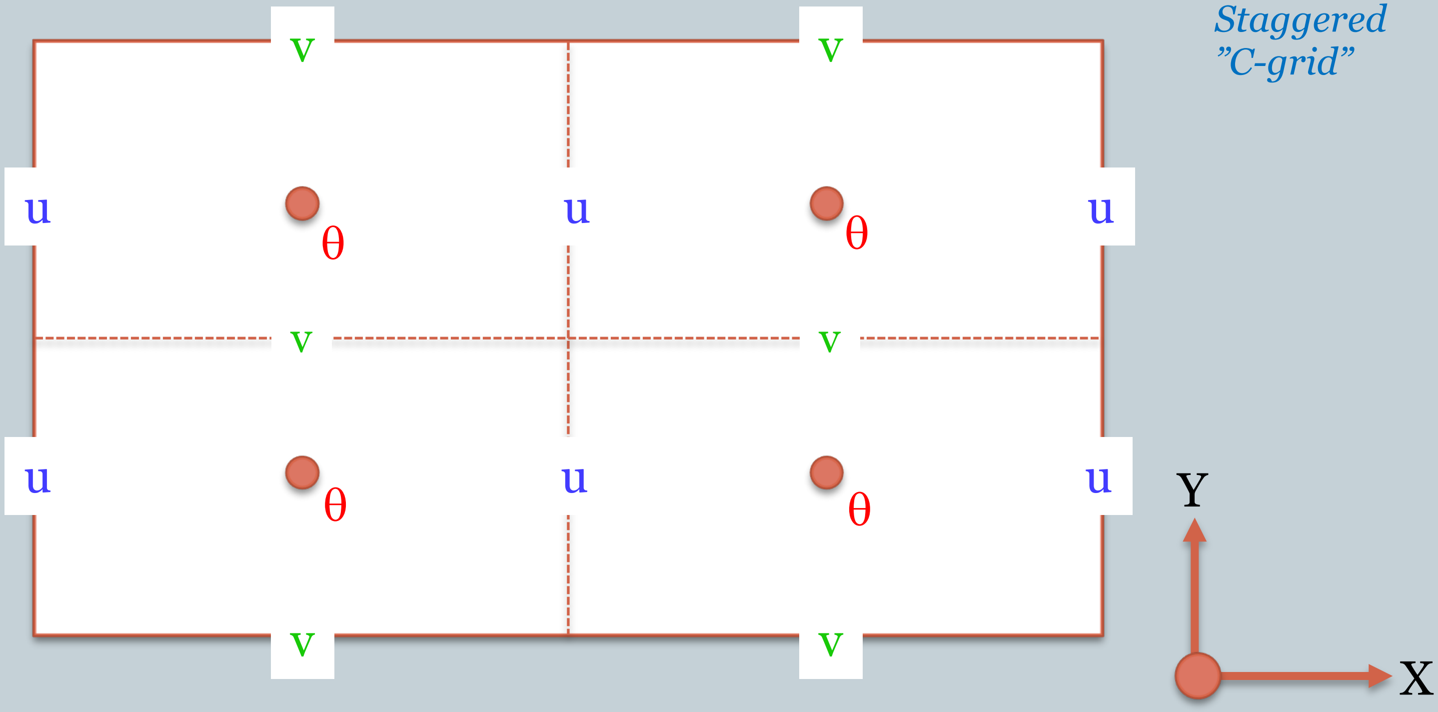
  - He uses an additional grid point (j-2) in the scheme.
    - Strongest damping for waves with worst phase speed errors
    - Coefficient of extra point is a free parameter
      - chosen to minimize the total error.
    - *Least total error* for $\alpha=(1+\mu)/6$ *(Figs. 5,7)*

Review: Takacs' error

- Error computation
  - **Total error** is mean square error (6.1)
  - **Dissipation error** (6.6)
  - **Dispersion error** (6.7) = (total - dissipation)
  - Calculations involve
    - standard deviation of true (q_T) and finite diff. (q_D) fields
    - linear (e.g. Pearson's) correlation coefficient ρ
      - If $\rho=1$, only error *is* due to dissipation *(amplitude)*
      - If $\rho=0$, only error *is* due to dispersion *(phase)*

$$E_{TOT} = \frac{1}{N}\sum_j (q_T - q_D)^2$$

Computer Program 2

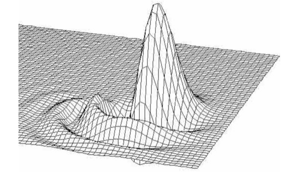


Staggered "C-grid"

Program 2 - recommendations

- Coding programs is like test taking …
  - There are distinct advantages to having a plan.
- 1) Make the necessary arrays 2D.
- 2) Code & evaluate the *initial conditions (ICs)*.
  - Create scalar field + U, V velocity components
  - Be careful with dimensions & physical locations
  - Plot it - plotting+code examples online *~tg457444/502/Pgm2*
- 3) Set the boundary conditions *(BCs)*
  - Alter your BC routine for two dimensions.
- 4) Now continue with 2D advection.

Program 2: Advection



- **Advection**
  - I set up 1-D arrays in my advection routines –
    - q1d(0:nx+1), u1d(nx+1), v1d(ny+1) *no ghost points for U, V !!*

- Advecting rows (X)
  - copy q1(i,j) to q1d
  - copy u(i,j) to u1d
  - pass q1d, u1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)

all j (rows)

- Advecting columns (Y)
  - copy q1(i,j) to q1d
  - copy v(i,j) to v1d
  - pass q1d, v1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)

all i (cols)

discuss: how many 2D q() arrays here?

Polar plots

REPRESENTING AMPLITUDE, PHASE ERROR

QUESTIONS WE ARE ADDRESSING:

1. **HOW DOES ERROR VARY WITH COURANT NUMBER?**
2. **HOW DOES ERROR VARY WITH WAVELENGTH?**

1. Find unit circle, courant #s

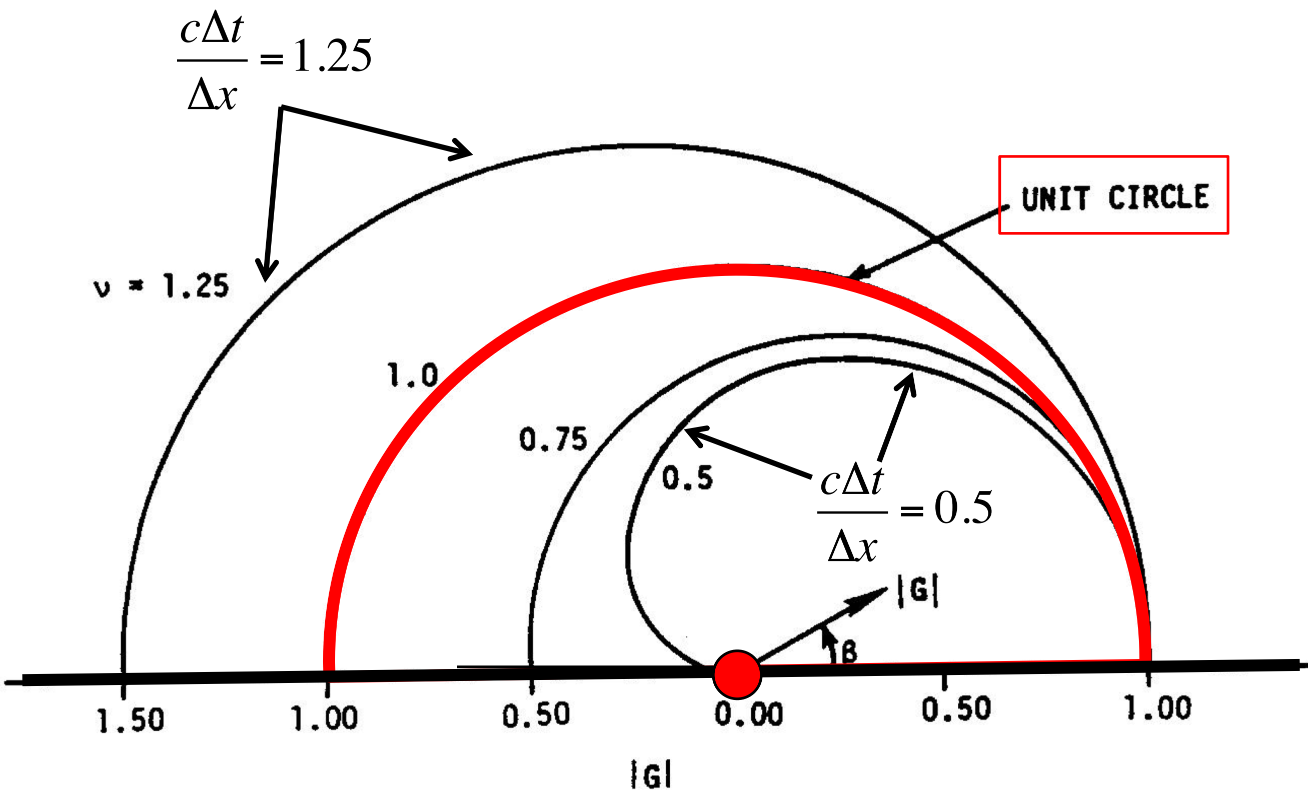


Figure 4-2 Amplification factor modulus for upstream differencing scheme.

Anderson et al., chapter 4

2. Find 2Δx, 4Δx, infinite waves

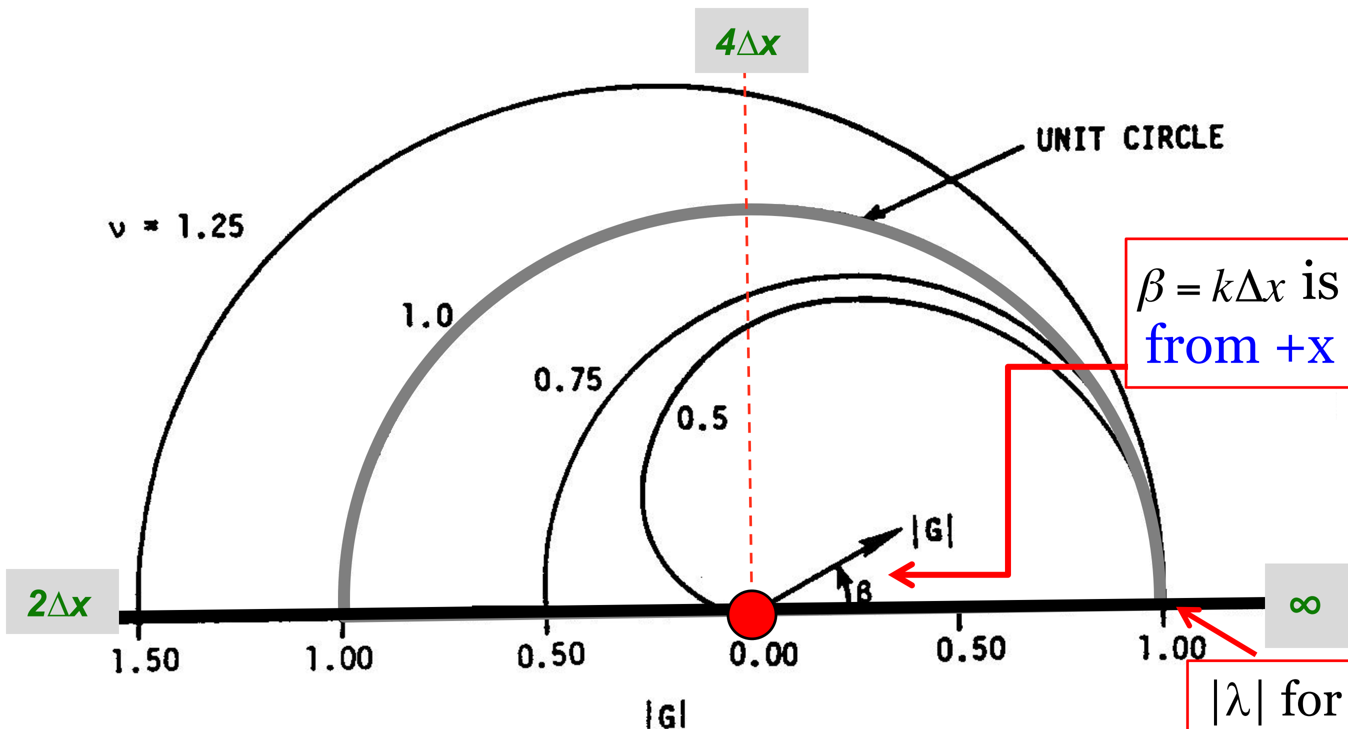


Figure 4-2 Amplification factor modulus for upstream differencing scheme.

$\beta = k\Delta x$ is angle from +x axis

$|\lambda|$ for *all* curves approach 1 for infinite wavelengths: no error!

3. Identify regions >1, <1

with respect to the unit circle!

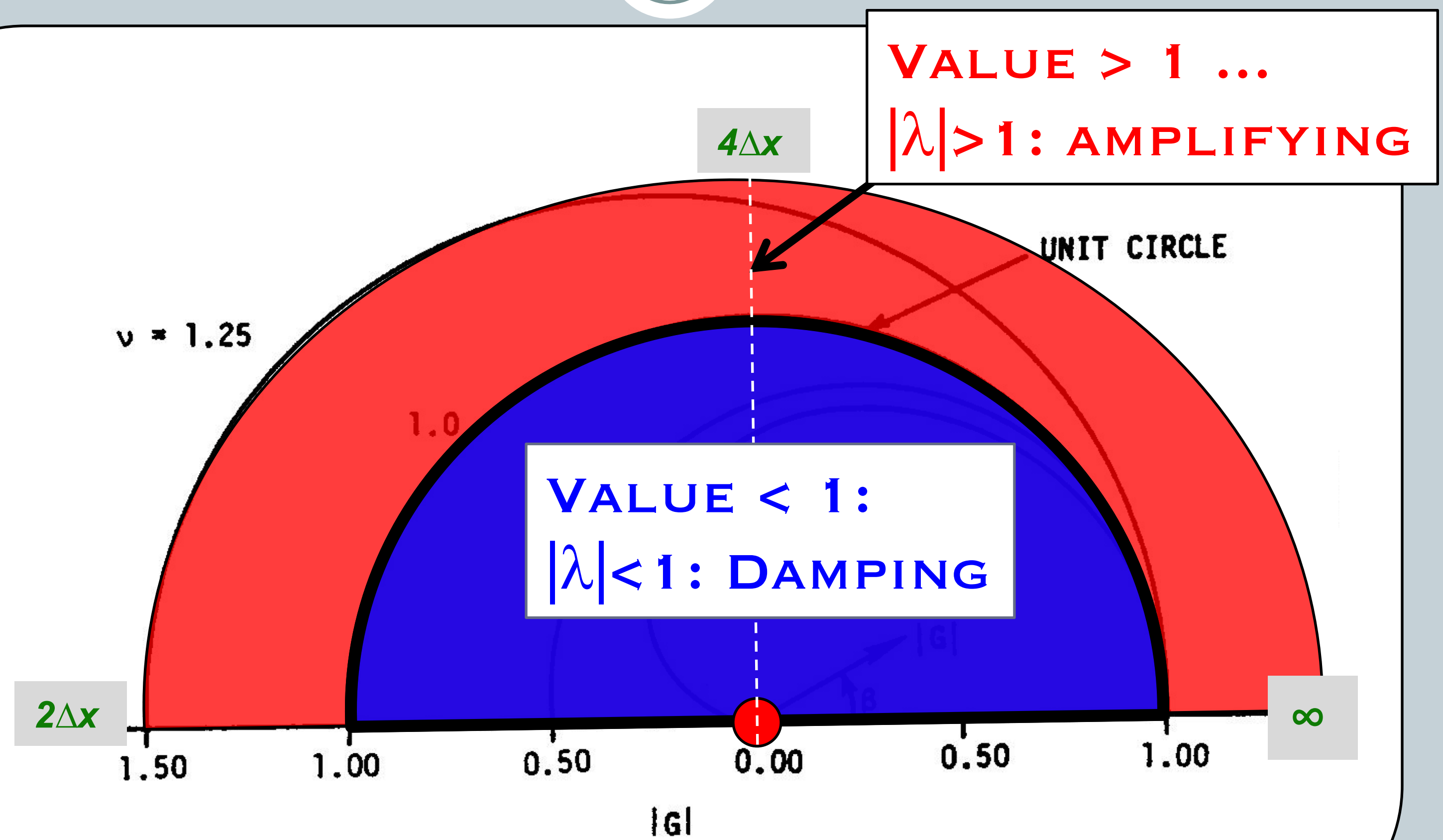
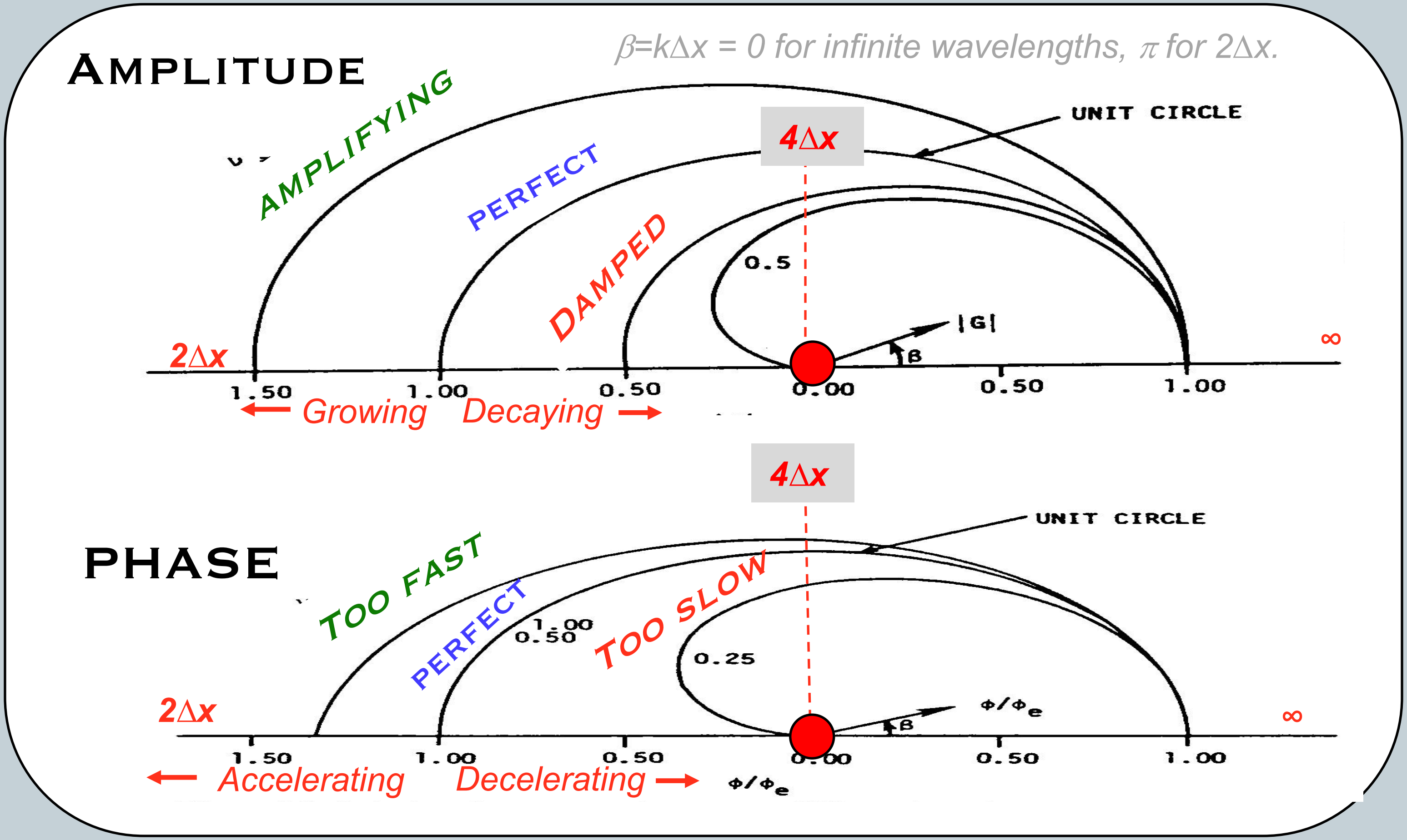


Figure 4-2 Amplification factor modulus for upstream differencing scheme.

Anderson et al., chapter 4

Review: Polar plots of errors



Anderson et al., chapter 4

Stability

OBJECTIVES:

DEVELOP THEORY AND METHODOLOGY FOR DETERMINING IF, HOW, AND WHEN A SCHEME HAS SATISFACTORY STABILITY.

References:
- A009 – Instability (physical)
- C015 – Instability (numerical)

Examples of instability in nature

- Thunderstorms

earthobservatory.nasa.gov/IOTD/view.php?id=78497 *smoke from Colorado fire*

- From
  - Aircraft
- Location
  - Wyoming
- Duration
  - hours
- Date
  - 7/11/2012
- Credit
  - DC3 project

Examples of instability in nature

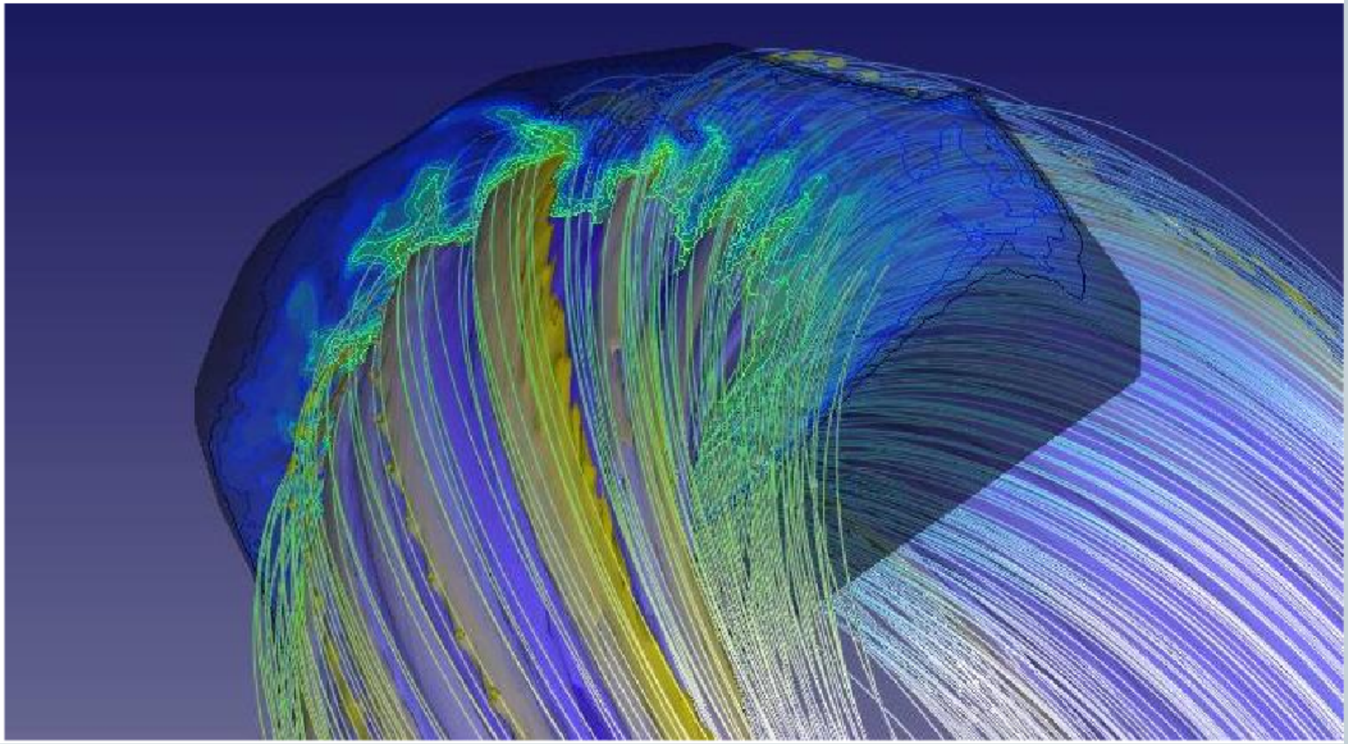
- Thunderstorms

- From
  - Ground
- Location
  - Booker TX
- Duration
  - hours
- Date
  - 6/3/2013
- Credit
  - Mike Oblinski

Instability has been defined as "outputs of internal states growing without bounds" or "if small perturbations cause changes that reinforce the original perturbation"

Examples of instability in nature

- Nonlinear instability in toroidal fusion plasma



- The *extended magnetohydrodynamics* (MHD) code M3D was used to study magnetic confinement and stability properties of fusion plasma in a Tokamak. *Edge Localized Modes* were noted, a new class of plasma instability.

- From:
  - Simulation
- Location:
  - Nat'l Energy Research Scientific Computing Center *(NERSC)*
- Credit
  - MIT: Linda Sugiyama

Ref: www.nersc.gov/science/fusion-science/a-new-class-of-tokamak-nonlinear-plasma-instability/

Stability: computational perspective

- What is stability?
  - Let's work backwards. What is *instability?*
- Instability
  - *Unstable numerical* scheme: numerical solution grows much more rapidly than the true one
- What is "more rapidly?"
  - We *must* be knowledgeable of the PDE properties (and thus of the physical phenomenon) to assess *reasonable* behavior.
  - If amplitude should *not* change –
    - any continued growth in the numerical solution is *unstable*
  - If exponential growth in amplitude is possible
    - than any growth *beyond* that is considered a numerical instability.

Quantifying instability

- Purpose: assess if F.D. scheme is stable.
  - Unstable scheme: numerical solution grows much more rapidly than the true one

---

- How do we do this?
  - First: consider any solution as Fourier series
    - *There are issues with discontinuities. This brought Fourier "much criticism" from the French Academy of Science at the time. We'll discuss this more later.*
  - Then: examine behavior of one wave component
    - if every Fourier component is stable …
    - i.e. *every possible wave's amplitude is bounded* …
    - then our scheme must be stable.

Stability

- Consider any solution as representable as a Fourier *series*
- Examine behavior of *one* component
  - if every Fourier component is stable …
  - i.e. *every possible wave's amplitude is bounded* …
  - then our scheme must be stable.
  - suppose our solution q(x,t) should *not* amplify *at all*. Then …

- Series: $$q_j^n = \sum_{k=-N}^{N} a_k^n e^{ikj\Delta x} \quad \Rightarrow \quad q_j^n = a_k^0 e^{ikj\Delta x}; \quad q_j^{n+1} = A_k \left(a_k^0 e^{ikj\Delta x} \right)$$

General series

Initial condition (superscript 0) for wavenumber "k"

Solution after *one* time step

Back up a minute: *math review*

- Some function f

$$f = e^{ikx}; \quad x = j\Delta x; \quad f = e^{ik(j\Delta x)}$$

- This has amplitude *1* and phase θ ..

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- So we're assuming f is some kind of sinusoidal function. Here, consider ***one*** wave component, with wavenumber ***k***.
- Wave*length* L = 2π/k so …
  - *kx has units of an angle*.

$$kx = 2\pi \frac{x}{L}$$

Amplification factor

- After one step the solution *amplitude* will be:

$$a_k^n = A_k a_k^{n-1}$$

- A_k is the **(numerical)** amplification factor for wavenumber k.

- We can relate the amplitude of wave number k to amplitudes at earlier times, back to $t=t_o$, assuming the physical solution is bounded:

$$a_k^n = A_k a_k^{n-1} = \ldots = \left[A_k\right]^n a_k^0$$

Stability condition

- The *von Neumann stability condition*:
  - The (numerical) amplification factor A_k of every resolvable Fourier component must be bounded such that: $$|A_k| \le 1 + \gamma\Delta t \quad ; \ \gamma \text{ independent of k,}\Delta\text{t,}\Delta\text{x}$$

Wikipedia

John von Neumann

  - We'll go with the more restrictive criteria: $|A_k| \le 1$
    - This "≤1" is satisfactory for *constant-speed advection* …for which there should be *no* distortion and *no* amplitude change w/time.
    - $|A_k| \le 1$ means the numerical solution results, over time, remain bounded by their initial values. Appropriate if the *norm* of the *true* solution is constant with time: $\|q^n\| \le \|q^0\|$

Measures of magnitude

- How we measure amplitude behavior: *norms*
  - For vector *x* - a chain of numbers -
    - L_1 norm: sum of absolute values of all numbers

$$\|x\|_1 \equiv \sum_{i=1}^{n} |x_i|$$

    - L_2 norm: square root of sum of squares

$$\|x\|_2 \equiv \sqrt{\sum |x_j|^2}$$

    - Infinite norm: maximum value:

$$\|x\|_\infty \equiv \max |x_i| \quad \text{for } 1 \leq \text{i} \leq \text{N}$$

  - There are similar relations for *matrix* norms

von Neumann's method: limitations

- *Small print:* $|A_k| \leq 1$
  - This is appropriate when the true solution is bounded by the norm of the initial data
  - If the stability criteria is met, every Fourier component is stable, and the full solution is, too.
  - The Von Neumann condition is a necessary and sufficient condition for stability.
    - The Von Neumann method is strictly speaking only applicable to linear, constant-coefficient problems
    - *Sufficiency* only for *single equations* in *one unknown*
    - *Periodic* boundary conditions are implied here.

Applying von Neumann's method

- Bottom line:
  - We'll insert a form

$$q_j^n = \tilde{q}^n e^{ikx} = \tilde{q}^n e^{ikj\Delta x}$$

  - Into the finite difference expression.

- The *spatial* information will be expressed in the exponential: $e^{ikj\Delta x}$.

- The *time* information will be included in the coefficient $\tilde{q}^n$, such that the *amplification factor* is:

$$A \text{ (or } \lambda) = \frac{\tilde{q}^{n+1}}{\tilde{q}^n}$$

$$(1-\cos x) = 2\sin^2\left(\frac{x}{2}\right)$$

*USEFUL TRIG IDENTITY!

Stability Example (1)

- The 1-D diffusion equation: $$\frac{q_j^{n+1} - q_j^n}{\Delta t} = K\frac{\left(q_{j+1}^n - 2q_j^n + q_{j-1}^n\right)}{\Delta x^2}$$
  - Substitute $q_j^n = \tilde{q}^n e^{ikj\Delta x}$, simplify:

Stability Example - *Summary*

- 1-D diffusion equation

$$\frac{q_j^{n+1} - q_j^n}{\Delta t} = K \frac{\left(q_{j+1}^n - 2q_j^n + q_{j-1}^n\right)}{\Delta x^2}$$

- Substituting $q_j^n = \tilde{q}^n e^{ikj\Delta x}$, and simplifying gives

$$\tilde{q}^{n+1} = \tilde{q}^n \left[1 + \sigma\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right] \quad (\sigma = \mathrm{K}\Delta \mathrm{t}/\Delta \mathrm{x}^2)$$

- and amplification factor *(now called λ)*:

$$\lambda = 1 + 2\sigma\left(\cos k\Delta x - 1\right) = 1 - 4\sigma \sin^2\left(\frac{k\Delta x}{2}\right)$$

- and from requiring $|\lambda \leq 1|$ we obtain: $0 \leq \sigma \leq {}^1/_2$

Operator definitions

USED FOR REMAINDER OF CLASS

See Durran, Appendix A.1

Operator definitions

- Shorthand for differencing:

$$\delta_{nx} f(x) = \frac{f\left(x + \frac{n\Delta x}{2}\right) - f\left(x - \frac{n\Delta x}{2}\right)}{n\Delta x}$$

$$\delta_x f(x) = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}$$

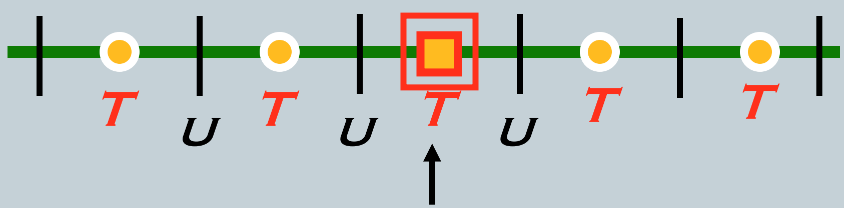
$$\delta_{2x} f(x) = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

- ... and for averaging:

$$\overline{f}^{nx} = \frac{1}{2}\left[f\left(x + \frac{n\Delta x}{2}\right) + f\left(x - \frac{n\Delta x}{2}\right)\right]$$

$$\overline{f}^{x} = \frac{f_{i+1/2} + f_{i-1/2}}{2}$$

$$\overline{f}^{2x} = \frac{f_{i+1} + f_{i-1}}{2}$$



Evaluate u•dT/dx here...

Durran (A.1), & Wilhelmson

Operators: manipulation

$$\delta_x^2 f = \delta_x\left(\delta_x f\right)$$

$$\delta_{\mathrm{x}}\left(\overline{u}^{x}\right) = \delta_{2x} u$$