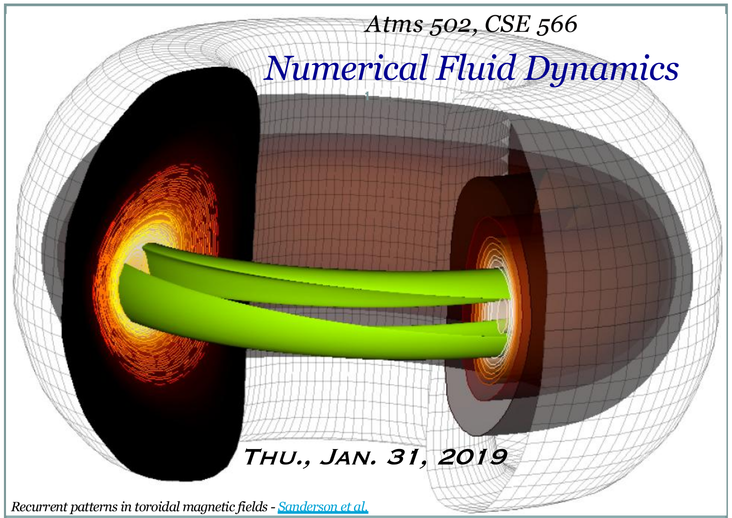
Atms 502, CSE 566

Numerical Fluid Dynamics



Thu., Jan. 31, 2019

Recurrent patterns in toroidal magnetic fields - Sanderson et al.

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ATMS 502
CSE 566

Thursday,
31 January 2019

Class #6

Plan for Today

- 1) REVIEW
  Takacs plots, method, error calc.
- 2) CODE/DATA:
  Program #2 - handout
- 3) NUMERICAL METHODS :
  Polar plots
  Stability
  Consider one harmonic …
  Norms
  von Neumann's method
  Apply to a numerical method
  Operator definitions

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Review: Takacs' plots

Takacs pp. 1053-1054

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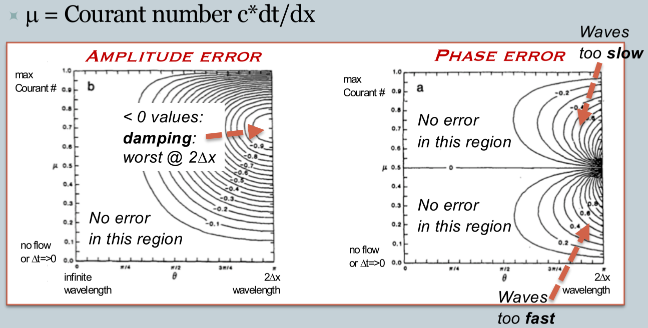
- Plots – Amplitude and Phase Error
  - θ = kΔx = nondimensional wavenumber
  - μ = Courant number c*dt/dx



ATMS 502 - Spring 2019 C022: Amplitude error; C023: Phase error; C026: Order of accuracy 1/31/19

Review: Takacs' scheme

Takacs pp. 1049-1056

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- Goal: balance dissipation, dispersion *(eqn 4.3)*
  - Chooses 2-step scheme for simplicity, cost
  - His method is 2nd order + "some of improved phase characteristics associated w/third-order scheme"

$$q_j^{n+1} = a_1 q_{j+1}^n + a_0 q_j^n + a_{-1} q_{j-1}^n + a_{-2} q_{j-2}^n$$

  - He uses an additional grid point (j-2) in the scheme.
    - Strongest damping for waves with worst phase speed errors
    - Coefficient of extra point is a free parameter
      - chosen to minimize the total error.
    - *Least total error* for α=(1+μ)/6 *(Figs. 5,7)*

ATMS 502 - Spring 2019 C022: Amplitude error; C023: Phase error; C026: Order of accuracy 1/31/19

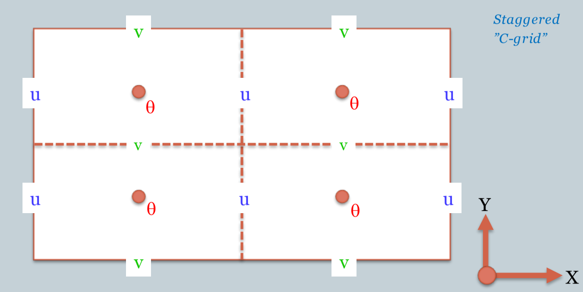
Review: Takacs' error

Takacs pp. 1055-1059

- Error computation
  - Total error is mean square error (6.1) $E_{TOT} = \frac{1}{N}\sum_j (q_T - q_D)^2$
  - Dissipation error (6.6)
  - Dispersion error (6.7) = (total - dissipation)
  - Calculations involve
    - standard deviation of true (q_T) and finite diff. (q_D) fields
    - linear (e.g. Pearson's) correlation coefficient ρ
      - If ρ=1, only error *is* due to dissipation *(amplitude)*
      - If ρ=0, only error *is* due to dispersion *(phase)*

ATMS 502 - Spring 2019 — *C022: Amplitude error; C023: Phase error; C024: Mean square error; C025: Correlation* — 1/31/19 — 5

Computer Program 2



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Program 2 - recommendations

- Coding programs is like test taking ...
  - There are distinct advantages to having a plan.
- 1) Make the necessary arrays 2D.
- 2) Code & evaluate the *initial conditions (ICs)*.
  - Create scalar field + U, V velocity components
  - Be careful with dimensions & physical locations
  - Plot it - plotting+code examples online *~tg457444/502/Pgm2*
- 3) Set the boundary conditions *(BCs)*
  - Alter your BC routine for two dimensions.
- 4) Now continue with 2D advection.

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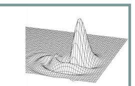
Program 2: Advection



- **Advection**
  - I set up 1-D arrays in my advection routines –
    - q1d(0:nx+1), u1d(nx+1), v1d(ny+1) *no ghost points for U, V !!*
- Advecting rows (X) *all j (rows)*
  - copy q1(i,j) to q1d
  - copy u(i,j) to u1d
  - pass q1d, u1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)
- Advecting columns (Y) *all i (cols)*
  - copy q1(i,j) to q1d
  - copy v(i,j) to v1d
  - pass q1d, v1d to *advect1d*
    - advect1d returns *q1d_out*
  - copy q1d_out to q1(i,j)

discuss: how many 2D q() arrays here?

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Polar plots

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REPRESENTING AMPLITUDE, PHASE ERROR

QUESTIONS WE ARE ADDRESSING:

1. HOW DOES ERROR VARY WITH **COURANT NUMBER?**
2. HOW DOES ERROR VARY WITH **WAVELENGTH?**

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1. Find unit circle, courant #s

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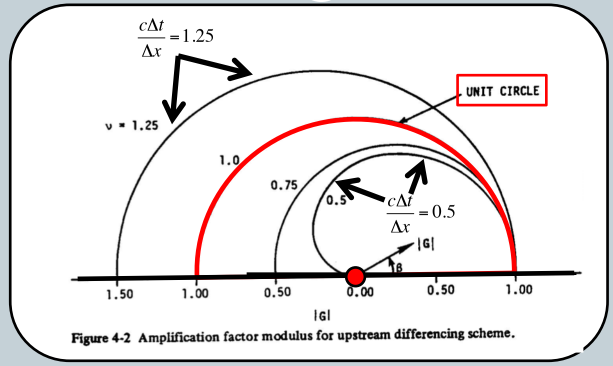

Figure 4-2 Amplification factor modulus for upstream differencing scheme.

Anderson et al., chapter 4

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2. Find 2Δx, 4Δx, infinite waves

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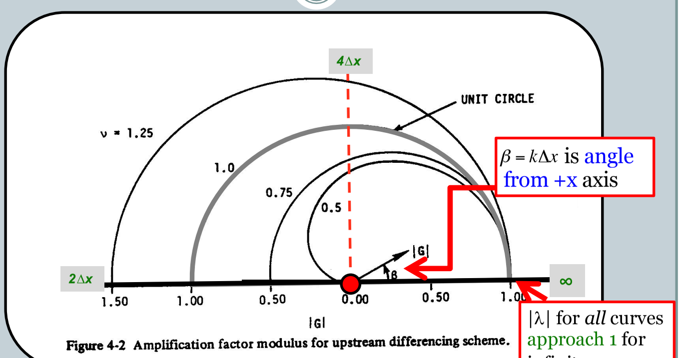

Figure 4-2 Amplification factor modulus for upstream differencing scheme.

|λ| for *all* curves approach 1 for infinite wavelengths: no error!

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3. Identify regions >1, <1 with respect to the unit circle!

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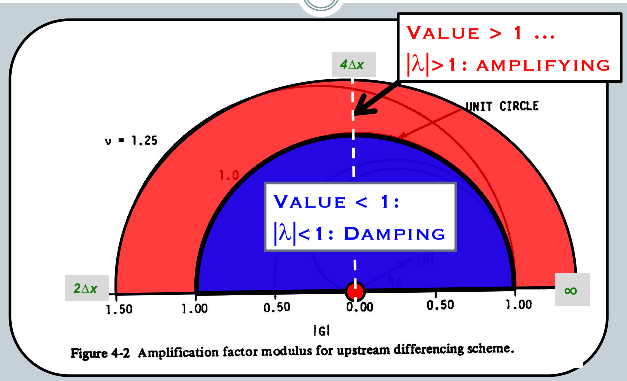

Figure 4-2 Amplification factor modulus for upstream differencing scheme.

Anderson et al., chapter 4

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Review: Polar plots of errors

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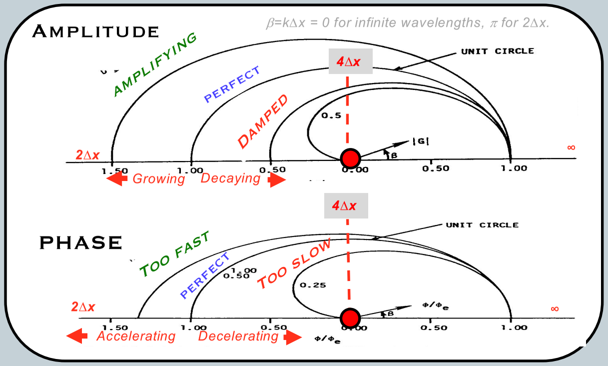

Anderson et al., chapter 4

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Stability

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OBJECTIVES:

DEVELOP THEORY AND METHODOLOGY FOR DETERMINING IF, HOW, AND WHEN A SCHEME HAS SATISFACTORY STABILITY.

References:
- A009 – Instability (physical)
- C015 – Instability (numerical)

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Examples of instability in nature

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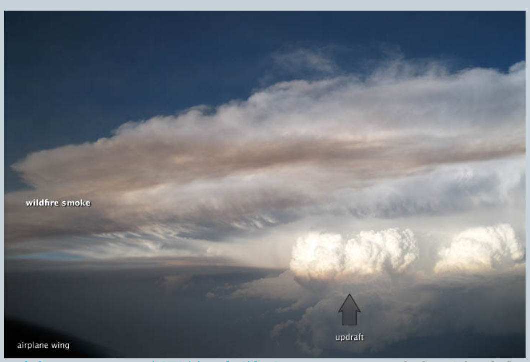
- Thunderstorms



earthobservatory.nasa.gov/IOTD/view.php?id=78497

smoke from Colorado fire

- From
  - Aircraft
- Location
  - Wyoming
- Duration
  - hours
- Date
  - 7/11/2012
- Credit
  - DC3 project

ATMS 502 - Spring 2019 A.009: Instability (physical) 1/31/19

Examples of instability in nature

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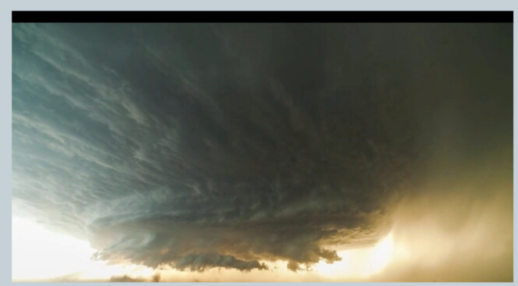
- Thunderstorms



- From
  - Ground
- Location
  - Booker TX
- Duration
  - hours
- Date
  - 6/3/2013
- Credit
  - Mike Oblinski

Instability has been defined as "outputs of internal states growing without bounds" or "if small perturbations cause changes that reinforce the original perturbation"

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Examples of instability in nature

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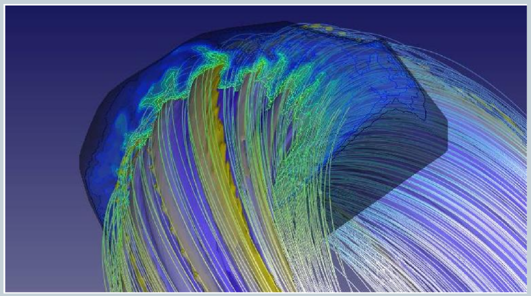
- Nonlinear instability in toroidal fusion plasma



- The *extended magnetohydrodynamics* (MHD) code M3D was used to study magnetic confinement and stability properties of fusion plasma in a Tokamak. *Edge Localized Modes* were noted, a new class of plasma instability.

- From:
  - Simulation
- Location:
  - Nat'l Energy Research Scientific Computing Center *(NERSC)*
- Credit
  - MIT: Linda Sugiyama

Ref: www.nersc.gov/science/fusion-science/a-new-class-of-tokamak-nonlinear-plasma-instability/

ATMS 502 - Spring 2019 — A.009: *Instability (physical)* — 1/31/19

Stability: computational perspective

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- What is stability?
  - Let's work backwards. What is *instability?*
- Instability
  - *Unstable numerical* scheme: numerical solution grows much more rapidly than the true one
- What is "more rapidly?"
  - We *must* be knowledgeable of the PDE properties (and thus of the physical phenomenon) to assess *reasonable* behavior.
  - If amplitude should *not* change –
    - any continued growth in the numerical solution is *unstable*
  - If exponential growth in amplitude is possible
    - than any growth *beyond* that is considered a numerical instability.

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Quantifying instability

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- Purpose: assess if F.D. scheme is stable.
  - Unstable scheme: numerical solution grows much more rapidly than the true one
- How do we do this?
  - First: consider any solution as Fourier series
    - *There are issues with discontinuities. This brought Fourier "much criticism" from the French Academy of Science at the time. We'll discuss this more later.*
  - Then: examine behavior of one wave component
    - if every Fourier component is stable …
    - i.e. *every possible wave's amplitude is bounded* …
    - then our scheme must be stable.

ATMS 502 - Spring 2019 — C.016: *Fourier Series* — 1/31/19

Stability

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- Consider any solution as representable as a Fourier *series*
- Examine behavior of *one* component
  - if every Fourier component is stable …
  - i.e. *every possible wave's amplitude is bounded* …
  - then our scheme must be stable.
  - suppose our solution q(x,t) should *not* amplify *at all*. Then …
- Series: $q_j^n = \sum_{k=-N}^{N} a_k^n e^{ikj\Delta x} \Rightarrow q_j^n = a_k^0 e^{ikj\Delta x}; \quad q_j^{n+1} = A_k \left(a_k^0 e^{ikj\Delta x} \right)$

GENERAL SERIES — INITIAL CONDITION *(SUPERSCRIPT 0)* FOR WAVENUMBER "K" — SOLUTION AFTER *ONE* TIME STEP

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Back up a minute: *math review*

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- Some function f

$$f = e^{ikx};\ \ x = j\Delta x;\ \ f = e^{ik(j\Delta x)}$$

  - This has amplitude *1* and phase θ ..

$$e^{i\theta} = \cos\theta + i\sin\theta$$

  - So we're assuming f is some kind of sinusoidal function. Here, consider *one* wave component, with wavenumber k.
  - *Wavelength* L = 2π/k so …

$$kx = 2\pi\frac{x}{L}$$

    - *kx has units of an angle.*

ATMS 502 - Spring 2019 | C.017: Euler's formula; C018: Complex numbers | 1/31/19

Amplification factor

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- After one step the solution *amplitude* will be:

$$a_k^n = A_k a_k^{n-1}$$

- A_k is the **(numerical)** amplification factor for wavenumber k.
- We can relate the amplitude of wave number k to amplitudes at earlier times, back to $t=t_o$, assuming the physical solution is bounded:

$$a_k^n = A_k a_k^{n-1} = \ldots = \left[A_k\right]^n a_k^0$$

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Stability condition

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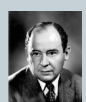
- The von Neumann *stability condition*:
  - The (numerical) amplification factor A_k of every resolvable Fourier component must be bounded such that:

$$\left|A_k\right| \le 1 + \gamma\Delta t\ \ ;\ \gamma \text{ independent of k,}\Delta\text{t,}\Delta\text{x}$$



  - We'll go with the more restrictive criteria: $\left|A_k\right| \le 1$
    - This "≤1" is satisfactory for *constant-speed advection* …for which there should be *no* distortion and *no* amplitude change w/time.
    - $|A_k| \le 1$ means the numerical solution results, over time, remain bounded by their initial values. Appropriate if the *norm* of the *true* solution is constant with time: $\|q^n\| \le \|q^0\|$

ATMS 502 - Spring 2019 | C.020: von Neumann stability condition | 1/31/19

Measures of magnitude

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- How we measure amplitude behavior: *norms*
  - For vector x - a chain of numbers -
    - L_1 norm: sum of absolute values of all numbers

$$\|x\|_1 = \sum_{i=1}^{n}|x_i|$$

    - L_2 norm: square root of sum of squares

$$\|x\|_2 = \sqrt{\sum |x_j|^2}$$

    - Infinite norm: maximum value:

$$\|x\|_\infty = \max|x_i|\ \ \text{for } 1 \le i \le N$$

  - There are similar relations for *matrix* norms

ATMS 502 - Spring 2019 | C.021: vector and matrix norms | 1/31/19

von Neumann's method: limitations

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- *Small print:* $|A_k| \le 1$
  - This is appropriate when the true solution is bounded by the norm of the initial data
  - If the stability criteria is met, every Fourier component is stable, and the full solution is, too.
  - The Von Neumann condition is a necessary and sufficient condition for stability.
    - The Von Neumann method is strictly speaking only applicable to linear, constant-coefficient problems
    - *Sufficiency* only for *single equations* in *one unknown*
    - *Periodic* boundary conditions are implied here.

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Applying von Neumann's method

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- Bottom line:
  - We'll insert a form

$$q_j^n = \tilde{q}^n e^{ikx} = \tilde{q}^n e^{ikj\Delta x}$$

  - Into the finite difference expression.
- The *spatial* information will be expressed in the exponential: $e^{ikj\Delta x}$.
- The *time* information will be included in the coefficient $\tilde{q}^n$, such that the *amplification factor* is: $A \text{ (or } \lambda) = \dfrac{\tilde{q}^{n+1}}{\tilde{q}^n}$

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Stability Example (1)

$(1-\cos x) = 2\sin^2\left(\frac{x}{2}\right)$

*USEFUL TRIG IDENTITY!

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- The 1-D diffusion equation: $\dfrac{q_j^{n+1} - q_j^n}{\Delta t} = K\dfrac{\left(q_{j+1}^n - 2q_j^n + q_{j-1}^n\right)}{\Delta x^2}$
  - Substitute $q_j^n = \tilde{q}^n e^{ikj\Delta x}$, simplify:

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Stability Example - *Summary*

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- 1-D diffusion equation

$$\frac{q_j^{n+1} - q_j^n}{\Delta t} = K \frac{\left(q_{j+1}^n - 2q_j^n + q_{j-1}^n\right)}{\Delta x^2}$$

- Substituting $q_j^n = \tilde{q}^n e^{ikj\Delta x}$, and simplifying gives

$$\tilde{q}^{n+1} = \tilde{q}^n \left[1 + \sigma\left(e^{ik\Delta x} - 2 + e^{-ik\Delta x}\right)\right] \quad (\sigma = K\Delta t/\Delta x^2)$$

- and amplification factor *(now called λ)*:

$$\lambda = 1 + 2\sigma(\cos k\Delta x - 1) = 1 - 4\sigma \sin^2\left(\frac{k\Delta x}{2}\right)$$

- and from requiring $|\lambda \le 1|$ we obtain: $0 \le \sigma \le {}^1/_2$

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Operator definitions

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USED FOR REMAINDER OF CLASS

ATMS 502 - Spring 2019 C032: Operator notation for finite differences 1/31/19

Operator definitions

See Durran, Appendix A.1

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- Shorthand for differencing:

$$\delta_{nx} f(x) = \frac{f\left(x + \frac{n\Delta x}{2}\right) - f\left(x - \frac{n\Delta x}{2}\right)}{n\Delta x}$$

$$\delta_x f(x) = \frac{f_{i+1/2} - f_{i-1/2}}{\Delta x}$$

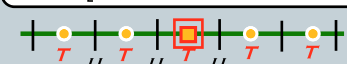
$$\delta_{2x} f(x) = \frac{f_{i+1} - f_{i-1}}{2\Delta x}$$

- ... and for averaging:

$$\overline{f}^{nx} = \frac{1}{2}\left[f\left(x + \frac{n\Delta x}{2}\right) + f\left(x - \frac{n\Delta x}{2}\right)\right]$$

$$\overline{f}^{x} = \frac{f_{i+1/2} + f_{i-1/2}}{2}$$

$$\overline{f}^{2x} = \frac{f_{i+1} + f_{i-1}}{2}$$



Evaluate u•dT/dx here...

Durran (A.1), & Wilhelmson

ATMS 502 - Spring 2019 C032: Operator notation for finite differences 1/31/19

Operators: manipulation

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$$\delta_x^2 f = \delta_x(\delta_x f)$$

$$\delta_x\left(\overline{u}^x\right) = \delta_{2x} u$$

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